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Integrated inventory model with controllable lead time involving investment for quality improvement in supply chain system

M. Vijayashree [a] * and R. Uthayakumar [a]

[a] *Department of Mathematics, The Gandhigram Rural Institute – Deemed University, Gandhigram, Dindigul, Tamil Nadu, India*

Abstract

The purpose of this article is to investigate a two-echelon supply chain inventory problem consisting of a single-vendor and a single-buyer with controllable lead time and investment for quality improvements. This paper presents an integrated vendor-buyer inventory model in order to minimize the sum of the ordering cost, holding cost, setup cost, investment for quality improvement and crashing cost by simultaneously optimizing the optimal order quantity, process quality, lead time and number of deliveries. Here the lead-time crashing cost has been assumed to be an exponentially function of the lead-time length. The main contribution of proposed model is an efficient iterative algorithm developed to minimize integrated total relevant cost for the single vendor and the single buyer systems with controllable lead time reduction and investment for quality improvements. It can be obtained simultaneously by optimizing the optimal solution, mathematical modelling and solution procedure are employed in this study for optimizing the order quantity, lead time, process quality and the number of deliveries from the vendor to the buyer in one production run with the objective of minimizing total relevant cost. Graphical representation is also presented to illustrate the proposed model. Numerical examples are presented to illustrate the procedures and results of the proposed algorithm. Matlab coding is also developed to derive the optimal solution and present numerical examples to illustrate the model.

Keywords: Integrated inventory model; vendor buyer coordination; and controllable lead time crashing cost; supply chain management; investment for quality improvements.

* Corresponding author email address: vijayashreemay30@gmail.com

1. Introduction

Some time ago, the companies can be obtained the competitive advantage by strengthening their competitiveness. As a result, we can use the integrated inventory model to obtain minimum the total relevant cost for both vendor and buyer. The just-in-time approach to productivity demands that small lots to be run in production. This can be only achieved if the setup time is reduced. The ability to reduce setup cost over time can be explained in the terms of the leaving curve.

Inventory control is important in supply chain management. Inventories play an extremely important role in a nation's economy. In recent years, most inventory problems have their focus on the integration between the vendor and the buyer. For supply chain management, establishing long-term strategic partnerships between the buyer and the vendor is advantageous for the two parties regarding costs, and therefore profits since both parties, to achieve improved benefits, cooperate and share information with each other. Therefore, several researchers [e.g., Amasaka (2002), Ben-Daya et al. (2004), Bylka (2003), Chang et al. (2006), Hoque et al. (2006), Ouyang et al. (2007), Pan et al. (2005), Villa (2001), Viswanathan (1998)] have shown that the buyer and the vendor can achieve their own minimal total cost, or increase their mutual benefit through strategic cooperation with each other.

In the production environment, lead time plays an important role in today's logistics management. Define as the time that elapses between the placements of an order into inventory Silver et al. (1988), lead time may influence customer service and impact inventory costs. As the Japanese example of just-in-time-production has shown, consequently reducing lead time may increase productivity and improve the competitive position of the company Tersine et al. (1995). Although lead time can be constant or variable, it is often treated as a prescribed parameter in most of the studies. Therefore, the lead time crashing cost function is a pricewise linear function [Liao et al. (1991), Ouyang et al. (2002), Ouyang et al. (1999)].

The number of advantages have been associated the efforts of control of the lead time (which is a goal of JIT inventory management philosophies that emphasizes high quality and keeps low inventory level and lead time to a practical minimum). Lead time management is a significant issue in production and operation management. In many practical situations, lead time can be reduced using an added crashing cost. In other words, lead time is controllable. The crashing of lead time mainly consists of the following components: order preparation, order transit, supplier lead time and delivery time Tersine (1994).

Supply chain management has taken a very important and critical role for any company in increasing globalization and competition in the market. A ***Supply Chain Model (SCM)*** is a network of suppliers, producers, distributors, and customers which synchronizes a series of interrelated business process in order to have (1) optimal procurement of raw materials from nature, (2) transportation of raw materials into a warehouse, (3) production of goods in the production centre, and (4) distribution of these finished goods to retailers for sale to the customers. With a recent paradigm shift to the ***Supply Chain (SC),*** the ultimate success of a firm may depend on its ability to link supply chain members seamlessly.

In the current ***Supply Chain Management (SCM)*** environment, companies are using JIT production to gain and maintain a competitive advantage. JIT requires a spirit of cooperation between the buyer and the vendor, and it has been shown that forming a partnership between the buyer and the vendor is helpful in achieving tangible benefits for both parties Goyal et al. (1992). In recent years, enterprises business models are different from the past due to globalization and information. The enterprises have to face many predicaments in internationalization, so there will be the emergence of the concept of supply chain. Compared to the traditional business model, the supply chain system can integrate the upstream and downstream companies between enterprises and use the resources more efficiently to create more profit.

The collaboration concept has become an accepted practice in many successful global business organizations and provides economic advantages for both a vendor and a buyer. As a certified supplier, the vendor needs to perfect production process by its efforts to improve the buyer's operational efficiency and maintain a win-win relationship with the buyer. The ultimate goal of JIT from the production/inventory management standpoint is to produce small sizes high quality products. Investing capital in shortening lead time and improving quality are regarded as the most effective means of achieving this goal. With such characteristics, researchers have modified traditional inventory models to incorporate the implementation of JIT concepts**.** ***Just-in-time (JIT)*** is a philosophy of manufacturing based on planned elimination of all wastes and on continuous improvement of productivity.

The issue of coordination in supply chain management (SCM) has received considerable attention from academic researchers and practitioners. Traditionally, both vendors and buyers in the supply chain system make decisions in search of their individual benefits. However, many researchers (e.g. Parlar et al. (1997), Qin et al. (2007), Sarmah et al. (2006), Weng (1997)] have pointed out that coordination between both parties is important in order to gain competitive advantages through cost reduction. The importance of coordination is further increased because vendors and buyers frequently implement the just-in-time (JIT) concept in their own systems. A recent study pointed out that coordination is crucial to successful JIT implementation for both parties Huang et al. (2004). A key technique in successful SCM is JIT application to multiple deliveries Chung et al. (2007) showed that increases in quality, productivity, and efficiency can be achieved through JIT delivery agreements. A recent study showed that if a long-term relationship has been established, both parties in the supply chain system can achieve further improved benefits through cooperation and information sharing Chang et al. (2006). Rau et al. (2008) presented a new integrated production-inventory policy that showed that the performance of integrated consideration is better than the performance of any independent decision from either the buyer or the vendor.

In this complex environment, successful companies have devoted considerable attention to reducing inventory cost and improving quality simultaneously. The return on investment for quality improvement is substantial and many papers have shown that improving quality could reduce waste, in other words, cut the cost. In addition, the probability of defects also makes a great impact on the inventory policy regarding production cycle and lot size, so it is important to always take quality issues into account for any business in a competitive supply chain environment nowadays. Therefore, in this present study integrated inventory model with controllable lead time involving investment for quality improvement in supply chain system and intend to proposes a simple solution procedure to search the optimal production, number of shipments and process quality that can minimise the integrated total relevant cost.

2. Review of the related literature

The single vendor single buyer integrated production inventory problem received a lot of attention in recent years. This renewed interest is motivated by the growing focus on supply chain management. Firms are realizing that a more efficient management of inventories across the entire supply chain through better coordination and more cooperation are in the joint benefit of all parties involved. Such collaboration is facilitated by the advances in information technology providing faster and cheaper communication means.

In real-life business environments, it is common to have a supplier who provides a product to its several retailer clients. In this type of supply chains, management is intended to figure out the best production-shipment policy in order to minimize the expected integrated system costs. Optimal inventory policies have been subject to a lot of research in recent years. In traditional ***Economic Order Quantity (EOQ)*** and ***Economic Production Quantity (EPQ)*** models, most of the most of research treating inventory problems, either in deterministic or probabilistic models, the stock out or setup costs is regarded as prescribed constants and equal at the optimum. However, the experience of the Japanese indicates that this need not be the case. In practice, setup cost may be controlled and reduced by virtue of various efforts, such as worker training, procedural changes, and specialized equipment acquisition. When inventory decisions in supply chains are made independently at each stage, they are usually based on the local inventory status and local performance objectives (local policies). These policies are simple to be defined and implemented, but ignore the implications that decisions at one stage can have on the others, let alone the fact that local objectives are often conflicting among each other, which often leads to sub optimize the Supply Chain (SC) performance.

In such cases, the ***Economic Lot Size (ELS)*** of one stage may not result in an optimal policy for the other stages. To overcome this problem, researchers have come up with a ***Joint Economic Lot Size (JELS)*** model where the ***Joint Total Relevant Cost (JTRC)*** for all stages has been optimized. Goyal (1976) first introduced an integrated inventory policy for a single vendor and a single purchaser. A supply chain is a system of facilities and activities that functions to procure, produce, and distribute goods to customers. Supply chain management is basically a set of approaches utilized to efficiently integrate suppliers, manufacturers, warehouses, and stores, so that merchandise is produced and distributed at the right quantities, to the right locations, and at the right time, in order to minimize system-wide costs (or maximize profits) while satisfying service level requirements Simchi-Levi et al. (2000).

In competitive environment, supply chain management has emerged as a popular production and logistics strategy for many contemporary firms, and the just-in-time (JIT) purchasing plays a crucial role in such supply chain environments. Companies are using JIT purchasing to gain and maintain a competitive advantage. The benefits of JIT purchasing include small lot sizes, frequent deliveries, consistent high quality, reduction in lead times, decrease in inventory levels, lower setup cost and ordering cost, and close supplier ties. In recent years, companies have found that there are substantial benefits from establishing a long-term sole-supplier relationship with supplier Martinich (1997). In the JIT environment, a close cooperation exists between supplier and purchaser to solve problems together, and thus maintains stable, long-term relationships. Supply chain is the sequence of business processes and activities from suppliers through customers that provide products, services and information to achieve customer satisfaction, i.e., a chain that can quickly respond to customer's requirement. Recently, the issue of just-in-time (JIT) manufacturing has received considerable attention, and one of the most novel issues are the integration of vendor and buyer in the supply chain system Chang et al. (2006).

Integration of different entities in the supply chain is an important way to gain competitive advantage and customer satisfaction. In current years, research dealing with inventory management in supply chain system has received attention from many scholars. Goyal (1976) is among the first researchers who studies integrated inventory model for single vendor single buyer system. He introduces a model for situation in which vendor produces a lot based on an infinite production rate and transfers it to the buyer by a lot-for-lot policy. He shows that making inventory decisions jointly among vendor and buyer can result in cost reduction compared to individual decisions.

The integration between vendor and buyer for improving the performance of inventory control has received a great deal of attention and the integrated approach has been examined for years. In 1986, Banerjee assumed that the vendor manufactures at a finite rate and considered a joint economic-lot-size model in which a vendor produces to order for a buyer on a lot-for-lot basis. Goyal (1988) relaxed the lot-for-lot policy and suggested that vendors economic production quantity should be an integer multiple of buyer purchase quantity. As a result of using the approach suggested in the Goyal's (1988) model, significant reduction in inventory cost can be achieved. Pan et al. (2002) improved Goyals (1988) model by considering lead time as a controllable factor in the model and obtained a lower joint total expected cost and shorter lead time. Inventory models incorporating lead time as a decision variable were developed by several researchers. Liao et al. (1991) addressed a probabilistic inventory model in which the lead time is a decision variable. Ben-Daya et al. (1994) extended Liao et al. (1991) model by allowing both the lead time and the order quantity as decision variables. Later, several researchers [see Ouyang et al. (1996), Moon et al. (1998), Ouyang et al. (2004), Ouyang et al. (2006), Ouyang et al. (2007), Jha et al. (2009)] investigated various integrated production-inventory models for lead time reduction in single-vendor single-buyer supply chain. Ha et al. (1997) proposed an integrated lot-splitting model of facilitating multiple shipments in small lots. Hoque et al. (2000) proposed an integrated production-inventory system involving the capacity of transport equipment. Yang et al. (2000) presented an integrated model considering economic ordering policy of deteriorated item. Nieuwenhuyse et al. (2006) found that lot splitting policies have benefited both the vendor and the buyer. Huang et. al. (2010) presented the permissible delay in payment problem in a single-vendor and a single-buyer integrated inventory model. Tasi (2011) developed a production and shipment model for a system that incorporates learning effect and deteriorating items and to derive an optimal joint total cost from the integrated perspective of both vendor and buyer. Uthayakumar et al. (2012) proposed a model that integrates the single vendor single buyer problem with order-processing cost reduction and process mean.

Lead time may influence customer service and impact inventory costs. As the Japanese example of just-in-time-production has shown, consequently reducing lead times may increase productivity and improve the competitive position of the company (see also Tersine et al. 1995). In most of researchers [See Abuo-El-Ata et al. (2002), Elwakeel et al. (2006), Hadley et al. (1963), Montgomery et al. (1973), Posner et al. (1972),Vijayan (2007)] dealing with inventory problems, either using deterministic or probabilistic models, The classical inventory models often assume lead time as a given parameter or a random variable which is not subject to control.

Traditionally, the lead time of inventory model is hypothesized as known or with certain probability distribution, which therefore is not subject to control. But in many practical situations, lead time can be reduced by an additional crashing cost. That is, it is controllable. In fact, Tersine (1982) thought that the lead time usually consists of the following components: order preparation, order transit, supplier lead time, delivery time, and set up time. In many practical situations, lead time can be reduced by an added crashing cost, in other words it is controllable. By shortening the lead time, we can lower the safety stock; reduce the loss caused by stock out. Decreasing lead time leads to the lower safety stock, reduction of the loss sales caused by stock out, improving the customer service level and increasing the competitive ability in business.

Controlling inventory is a process and a method of total inventory management. Time-based competition focuses on the reduction of overall system response time and inventory lead time reduction has been one of favourite topics for both researchers and practitioners Pan et al. (2005). Liao et al. (1993) first presented a stochastic inventory model with lead time being the variable. Ben-Daya et al. (1994) modified Liao et al. (1993) by including both lead time and order quantity as decision variables. Ouyang et al. (1996) extended Ben-Daya et al. (1994) model by allowing shortages and treated the stockout treated of backorders and lost sales. Ouyang et al. (1998) developed a minimax distribution free procedure for mixed inventory model with variable lead time. Pan et al. (2001) modified Ouyang et al. model (1996) by considering back-order discount. Pan et al. (2002a) assumed the crash cost is a function of both the order quantity and the reduced lead time, and then established inventory models with fixed and variable lead time crash cost. Pan et al. (2005) investigated an integrated inventory system in which shortage is allowed and both lead time and backordering are negotiable. Chang et al. (2006) proposed integrated vendor-buyer cooperative inventory models with controllable lead time and ordering cost reduction. Vijayashree et al. (2014) developed an integrated inventory model with controllable lead time and setup cost reduction for both non-defective items. Priyan et al. (2014) developed mathematical modelling for EOQ inventory system with advance payment and fuzzy parameters.

Quality has been highly emphasized in modern production/inventory management systems. Also, it has been evidenced that the success of Just-In-Tim (JIT) production is partly based on the belief that quality is a controllable factor, which can be improved through various efforts such as worker training and specialized equipment acquisition. In the classical inventory model, it is implicitly assumed that the quality level is fixed at an optimal level, i.e., all items are assumed to have perfect quality. However, in the real production environment, it can often be observed that there are defective items being produced due to imperfect production processes. The defective items must be rejected, repaired, reworked, or, if they have reached the customer, refunded. In all cases, substantial costs are incurred. Therefore, for the system with an imperfect production process, the manager may consider investing capital on quality improvement, so as to reduce the quality-related costs.

In the inventory literature, Porteus (1985) first introduced the concept and developed a framework for investing in reducing EOQ model set-up cost. Then, Ouyang et al. (2004) investigated the influence of ordering cost reduction on modified continuous review inventory systems involving variable lead time with partial backorders. Hong et al. (1995) presented a model including a budget constraint and other types of continuous functions for quality enhancement and setup cost reduction. Ouyang et al. (2000) investigated the impact of quality improvement on the modified lot size reorder point models involving variable lead time and partial backorders. Ouyang et al. (2002) extended Ouyang et al. (2000) model by investing in process quality improvement and setup cost reduction simultaneously. Later, many researchers [See Billngton (1987), Kim et al. (1992), Annadurai et al. (2010), Coates (1996)] developed EPQ models with ordering/setup cost reduction. Uthayakumar et al. (2013) developed supply chain model with variable lead time under credit policy.

The relation between the quality and inventory reduction is critical for both practitioners and academics because numerous modern production systems advocate reduction in inventory and improvement in quality. For example, Voss (1987) claims that just-in-time production systems lead to increased quality and reduced inventory. In addition, Kekre et al. (1992) shows that there exists a negative relationship between inventory and quality based on empirical results. Vijayashree et al. (2014) developed a two-stage supply chain model with selling price dependent demand and investment for quality improvement. Ouyang et al. (2007) developed an integrated vendor-buyer inventory model with quality improvement and lead time reduction. Yang et al. (2004) developed an integrated inventory model involving deterministic variable lead time and quality improvement investment. We assume that crashing cost is linear function and lead time is weeks.

Today's supply chain environment requires a new spirit of cooperation between the single-buyer and the single-vendor. We consider integrated inventory model with controllable lead time involving investment for quality improvement in supply chain system. The lead time is identical for all buyers as well as it can be shortened by paying an additional crashing cost which is exponentially function of lead time. And lead time expressed in weeks. The objective of this paper is to find out an optimal inventory strategy that can minimize the value of the integrated total relevant cost for both the single vendor and the single buyer. Finally, a numerical example is presented to illustrate the proposed model.

The study also takes accounts of the following aspects: In section 3, the fundamental notations and assumptions of this study is provided. Section 4 describes the model development. In section 5 an efficient algorithm is developed to obtain the optimal solution. A numerical example is provided in section 6 to illustrate the results. Finally, conclusions are shown and suggestions for future research are given in Section 7.

3. Notations and assumptions

First of all, the following notations and assumptions are used throughout this paper to develop the proposed model.

a. Notations

To establish the proposed model, the following notations are used:

D Buyer's expected demand rate in units per unit time

P Vendor's production rate in units per unit time, $P > D$

Q Buyer's order quantity in units

A Buyer ordering cost per order

S Vendor's setup cost per setup

L_s Normal duration to arrive the items in buyer inventories

L_e Minimum duration to arrive the items in buyer inventories

L Length of lead time

c_v Unit production cost paid by the vendor

c_b Unit purchase cost paid by the buyer

n The number of shipments in which the product is delivered from the vendor to the buyer in one production cycle, a positive integer, a decision variable.

r Annual inventory holding cost per dollar invested in stocks

s Vendor unit defective cost per defective item.

θ Probability of the vendor's production process that can go out-of-control.

θ_0 Original probability of the vendor's production process that can go out-of-control.

$I(\theta)$ Vendor's capital investments require for reducing the out of control probability form θ to θ_0 .

i Vendor's fractional opportunity cost of capital per unit time.

TRC Total Relevant cost for the single vendor and single the buyer.

b. Assumptions

To develop the proposed model, we adopt the following assumptions:

1. The system consists of a single vendor and a single buyer for a single product in this model, and the inventory system deals with only one type item.
2. The buyer orders a lot of size Q and the vendor manufactures nQ with a finite production rate P $(P > D)$ at one set-up, but ship quantity Q to the buyer over n times. The vendor incurs a set-up cost S for each production run and the buyer incurs an ordering cost A for each order of quantity Q.
3. There is vendor and buyer for a single product in this model.
4. The demand X during lead time L follows a normal distribution with mean μL and standard deviation $\sigma\sqrt{L}$.
5. The inventory is continuously reviewed. The buyer places the order when the on hand inventory reaches the reorder point R .
6. The buyer places the order when the inventory position reaches the reorder point R. The reorder point R = the expected demand during lead time + safety stock, that is, $R = DL + k\sigma\sqrt{L}$ where k is a safety factor and σ is the standard deviation.
7. The extra cost incurred by the vendor will be transferred to the buyer if shortened lead time is requested.
8. If the buyer is not eager to add extra cost to control the lead time, he should obtain his items at exactly normal lead time (L_s) and crashing cost is zero. Here, the buyer added crashing cost to control the lead time. Therefore, the buyer lead time L should be within this interval $L \in [L_e, L_s]$ that is $L_e \le L < L_s$.
9. The crashing costs were observed to grow with lead time by a proportion which can be approximated by an exponentially function of lead time. Therefore, the lead-time crashing cost per order $R(L)$, is assumed to be an exponentially function of L and is defined as $R(L) = \begin{cases} 0 & if \ \ \mathrm{L} = L_s \\ e^{C/L}, & \text{if } L_e \le L < L_s \end{cases}$ where C is a positive constant and L_e, L_s represents the minimum and the normal lead times respectively.
10. The relationship between lot size and quality is formulated as follows: while vendor is producing a lot, the process can go out of control with a given probability θ each time another unit is produced. The process is assumed to be in control in the beginning of the production process. Once out of control, the process produces defective items and continues to do so until the entire lot is produced. (This assumption is in line with Porteus (1986)).
11. The out-of-control probability θ is a decision variable, and is illustrated by a logarithmic investment function. The quality improvement and capital investment is illustrated by $q(\theta) = q\ln(\theta_0/\theta)$ for $0 < \theta \le \theta_0$, where θ_0 is the current probability that the production process can go out of control, and $q = 1/\xi$ with ξ meaning the percentage decrease in θ per dollar increase in $q(\theta)$. The application of the logarithmic function on capital investment and quality improvement has been proposed by many authors, for example, Porteus (1986); Hong et al. (1995); Ouyang et al. (2000) Yang et al. (2004) and Ouyang et al. (2006).

4. Model development

a. Integrated Total Cost (ITC)

The joint total expected cost per unit time in Pan et al. (2002) is the sum of the following elements
Ordering cost per unit time = $A/Q/D = AD/Q$

Buyer's holding cost per unit time is $=\left(\frac{Q}{2}+k\sigma\sqrt{L}\right)rc_b$

Lead time crashing cost per unit time= $(D/Q)R(L)$

Vendor setup cost per year $=\left(\frac{D}{nQ}\right)S$

Vendor's holding cost per unit time: vendor's average inventory is evaluated as the difference of the vendor's accumulated inventory and the buyer's accumulated inventory (see Figure 1).

That is $\left\{\left[nQ\left(\frac{Q}{P}+(n-1)\frac{Q}{D}\right)-\frac{n^2Q^2}{2P}\right]-\left[\frac{Q^2}{D}(1+2+\ldots+(n-1))\right]\right\}\frac{D}{nQ}$

$=\frac{Q}{2}\left[n\left(1-\frac{D}{P}\right)-1+\frac{2D}{P}\right]$

So the vendor's holding cost per unit time is $rc_v\frac{Q}{2}\left[n\left(1-\frac{D}{P}\right)-1+\frac{2D}{P}\right]$

Accordingly, the integrated total cost per unit time for the single vendor and the single buyer integrated inventory system is given by

$$ITC(Q,L,n)=\frac{D}{Q}\left(A+\frac{S}{n}+R(L)\right)+\frac{Q}{2}r\left(\left(n\left(1-\frac{D}{P}\right)-1+\frac{2D}{P}\right)c_v+c_b\right)+rc_bk\sigma\sqrt{L} \tag{1}$$

Defective item rework cost per unit time: the expected number of defective items in a run of size nQ with a given probability of θ that the process can go out of control is $\frac{n^2Q^2\theta}{2}$ (see Porteus(1986) for detail derivation). Thus, the defective cost per unit time is given by $\frac{snQD\theta}{2}$

Hence, the total cost incorporating the defective cost per year can be represented by

$$TC(Q,L,n)=ITC(Q,L,n)+\frac{snQD\theta}{2} \tag{2}$$

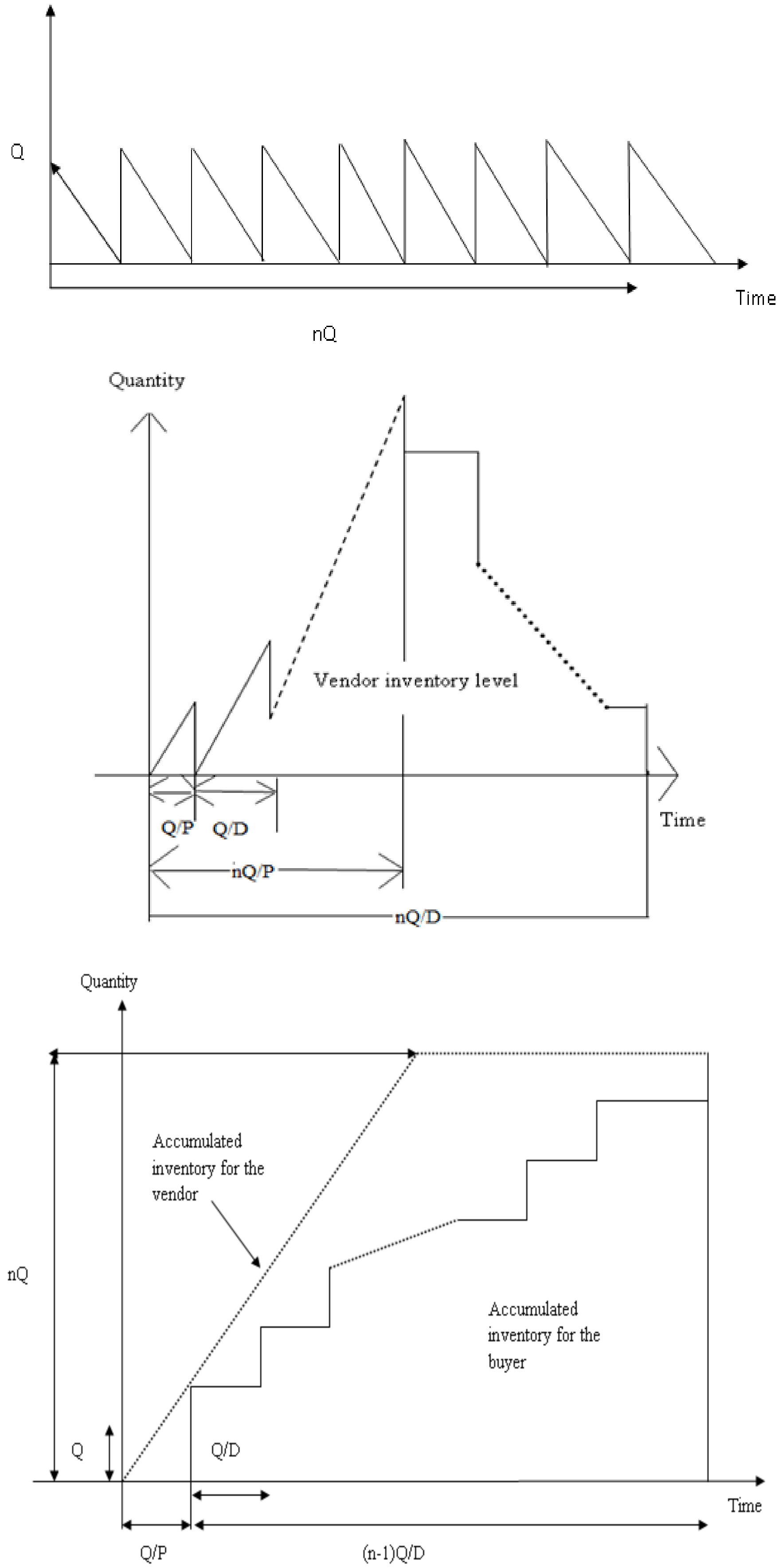

Figure 1. The inventory pattern for the buyer and vendor

b. Investment for quality improvement

Based on equation (2), this study is an attempt to study the effect of investment on quality improvement. Consequently, the objective of the integrated model is to minimize the sum of the ordering/setup cost, holding cost, quality improvement and crashing cost by simultaneously determining the optimal values of Q, n, and L, subject to the constraint that $0 < \theta \le \theta_0$. Thus, the total relevant cost per year is

$$TRC(Q, L, n, \theta) = TC(Q, L, n) + iq \ln \frac{\theta_0}{\theta}$$

$$\frac{D}{Q}\left(A + \frac{S}{n} + R(L)\right) + \frac{Q}{2} r\left(\left(n\left(1 - \frac{D}{P}\right) - 1 + \frac{2D}{P}\right)c_v + c_b\right) + rc_b k\sigma\sqrt{L} + \frac{snQD\theta}{2} + iq\ln\frac{\theta_0}{\theta} \quad (3)$$

$0 < \theta \le \theta_0$ where i is the fractional opportunity cost of capital per unit time.

Therefore, the problem under study can be formulated as the following nonlinear programming model.

Minimize $TRC(Q, L, n, \theta)$

$$= \frac{D}{Q}\left(A + \frac{S}{n} + R(L)\right) + \frac{Q}{2}\left(\left(n\left(1 - \frac{D}{P}\right) - 1 + \frac{2D}{P}\right)rc_v + rc_b + snD\theta\right) + rc_b k\sigma\sqrt{L} + iq\ln\frac{\theta_0}{\theta} \quad (4)$$

Subject to $0 < \theta \le \theta_0$

In order to find the minimum cost for this non-linear programming problem, ignore the constraint $0 < \theta \le \theta_0$ for the moment and minimize the total relevant cost function over Q, θ and L with classical optimization techniques by taking the first partial derivatives of $TRC(Q, L, n, \theta)$ with respect to Q, θ and $L \in [L_e, L_s]$ as follows:

$$\frac{\partial TRC(Q, L, n, \theta)}{\partial Q} = -\frac{D}{Q^2}\left(A + \frac{S}{n} + R(L)\right) + \frac{r}{2}\left(\left(n\left(1 - \frac{D}{P}\right) - 1 + \frac{2D}{P}\right)c_v + c_b + snD\theta\right) \quad (5)$$

$$\frac{\partial TRC(Q, L, \theta, n)}{\partial \theta} = \frac{snDQ}{2} - \frac{iq}{\theta} \quad (6)$$

$$\frac{\partial TRC(Q, L, n, \theta)}{\partial L} = -\frac{DCe^{\frac{C}{L}}}{QL^2} + \frac{1}{2} rc_b k\sigma L^{-\frac{1}{2}} \quad (7)$$

By setting Eq. (5) and (6) equal to zero, for a given value of $L \in [L_e, L_s]$, we obtain

$$Q = \sqrt{\frac{2D\left(A + \frac{S}{n} + e^{\frac{C}{L}}\right)}{r\left(\left(n\left(1 - \frac{D}{P}\right) - 1 + \frac{2D}{P}\right)c_v + c_b\right) + snD\theta}} \tag{8}$$

$$\theta = \frac{2iq}{snDQ} \tag{9}$$

Theoretically, for fixed n and $L \in [L_e, L_s]$ we can obtain the values of Q^*, θ^*. Moreover, it was found that the second order sufficient conditions are satisfied as follows. For fixed n the Hessian matrix $TRC(Q, L, \theta, n,)$ is positive definite Q^*, θ^* and L^*. The proof is shown in the appendix.

The subsequently algorithm is proposed to find the optimal value of order quantity Q, process quality θ, Lead time L, number of deliveries n.

5. Algorithm

Step1. Let $n = 1$. Since n is integer and set $\theta = \theta_0$.

Step2. Perform step (2.1) and (2.6) for all integer values of L in this interval $[L_e, L_s]$.

2.1. Use n, θ to compute Q from equation (8).
2.2. Use Q and n to compute θ from equation (9).
2.3. Repeat steps (2.1)-(2.2) until no change occurs in the values of Q and θ. Denote by Q^* and θ^*, respectively.
2.4. If $\theta^* < \theta_0$, then the solution is optimal for given $L \in [L_e, L_s]$. Denote the solution by $(Q^* \theta^*, L^*)$.
2.5. If $\theta^* \geq \theta_0$, then take $\theta^* = \theta_0$, and utilize equation (8) to determine new Q^* similar to the one in (2.1). The result is denoted by (Q^*, θ^*).
2.6. Compute the corresponding $TRC(Q, L, n, \theta)$, by putting Q, θ in equation (4)

Step3. Let $TRC(Q^*, \theta^*, L^*, n)$=minimum of $TRC(Q, L, n, \theta)$, then (Q^*, L^*, θ^*) is an optimal solution for fixed n.

Step4. Set $n = n + 1$ repeat steps (2)-(3) get $TRC(Q^*, L^*, \theta^*, n,)$.

Step5. If $TRC(Q^*, L^*, \theta^*, n) \leq TRC(Q^*, L^*, \theta^*, n-1)$; go to step 4, otherwise go to step 6.

Step6. $TRC(Q^*, \theta^*, n^*, L^*) = TRC(Q^*, \theta^*, n-1, L^*)$, then $(Q^*, L^*, n^*, \theta^*)$ is optimal solutions.

6. Numerical example

In this section, a numerical example is given to illustrate the above solution procedure. We consider the numerical example with the following data $D = 600$ unit/year, $P = 2000$ unit/year, $c_b = 100/\text{unit}$, $A = \$200/\text{order}$, $S = \$1500/\text{setup}$, $\text{r} = 0.2$, $c_v = 70/\text{unit}$, $\text{k} = 1.31$, $\sigma = 7$ unit/week, Crashing Cost $R(L) = \begin{cases} 0 & \text{if L} = 6 \\ e^{\frac{C}{L}} & \text{if } 1 \le L < 6 \end{cases}$ i.e., where $C = 5$ minimum lead time $L_e = 1$ week normal lead time $L_s = 10$ week. Applying the solution procedure of the proposed algorithm, the computational results are demonstrated in Table (1). The optimal solutions from Table (1) can be read off as optimal lead time $L^* = 2$ weeks, Order quantity $Q^* = 139$ units, number of shipments $n^* = 3$, process quality $\theta^* = 0.000021316$ and corresponding minimum total relevant cost $TRC^* = 6507$. Graphical representations are shown Figures (2) and (3).

Table 1. Optimal solutions for different values of lead time

	, where C=5, $1 \le L < 6$ $R(L) = e^{\frac{C}{L}}$								
n	**L=1**			**L=2**			**L=3**		
	Q	**θ**	**TRC**	**Q**	**θ**	**TRC**	**Q**	**θ**	**TRC**
1	301	0.000029531	7627	290	0.000030651	7426	289	0.000030757	7470
2	196	0.000022676	7005	183	0.000024287	6649	182	0.000024420	6685
3	152	0.000019493	6994	**139**	**0.000021316**	**6507**	138	0.000021471	6536
4	127	0.000017498	7142	114	0.000019493	6538	113	0.000019666	6560
5	110	0.000016162	7348	98	0.000018141	6638	97	0.000018328	6654

Table 1. Continued

	, where C=5, $1 \le L < 6$ $R(L) = e^{\frac{C}{L}}$						$R(L) = 0$		
n	**L=4**			**L=5**			**L=6**		
	Q	**θ**	**TRC**	**Q**	**θ**	**TRC**	**Q**	**θ**	**TRC**
1	289	0.000030757	7515	281	0.000031746	7557	289	0.000030757	7590
2	182	0.000024420	6728	182	0.000024420	6769	182	0.000024420	6799
3	138	0.000021471	6577	138	0.000021471	6617	138	0.000021471	6644
4	113	0.000019666	6600	113	0.000019666	6639	113	0.000019666	6634
5	97	0.000018328	6692	97	0.000018328	6730	97	0.000018328	6753

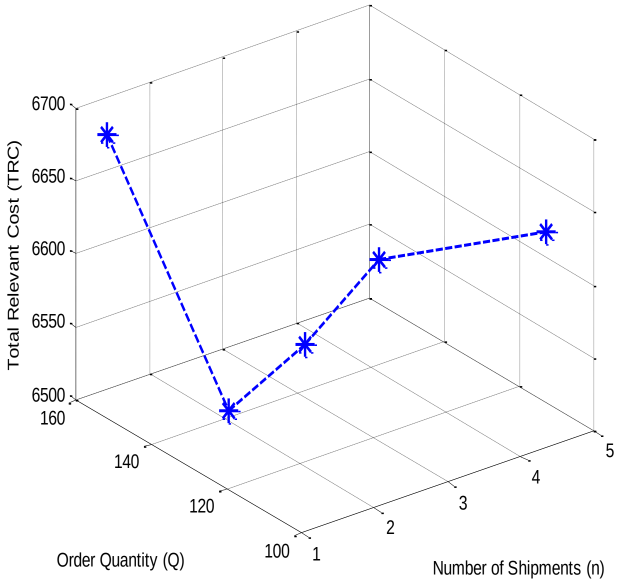

Figure 2. Graphical representations for optimal solution for TRC when $L^* = 2$, $n^* = 3$.

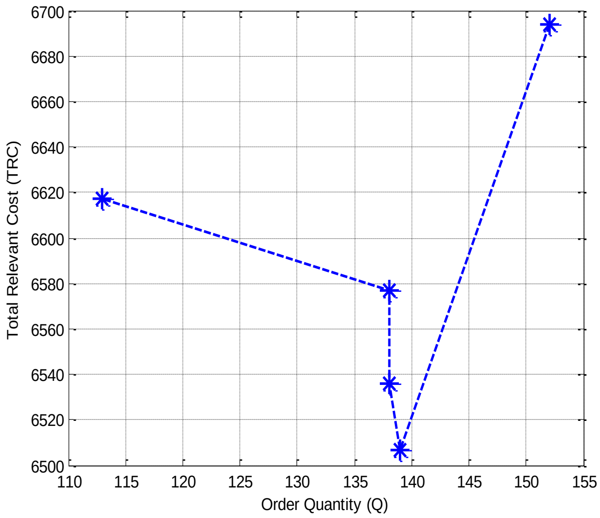

Figure 3. Graphical representation of the optimal solution.

7. Conclusion

In this paper, an inventory model for two-stage supply chain is investigated. A supply chain with single vendor and single buyer is considered. In this study, we consider integrated inventory model with controllable lead time involving investment for quality improvement in supply chain system. Lead time is an important element in any inventory management system. Industrial buyers often call it as lead time. Firm can shorten delivery times by storing inventory or having excess capacity. In many practical situations, lead time can be reduced at an added crashing cost; in other words, it is controllable. Here, the buyers lead time can be shortened by paying an additional crashing cost which is exponentially function of lead time.

In our model, the capital investment in quality improvement is assumed to be a logarithmic function. The main contribution of this proposed model is an efficient iterative algorithm has been developed to minimize the total relevant cost for the single vendor and single buyer integrated system with investment for quality improvement by simultaneously optimizing the optimal order quantity, lead time, process quality and number of shipments from the single vendor to the single buyer in a production cycle. A solution procedure is developed to find the optimal solution. A computer code using the software Matlab is developed to derive the optimal solution of the system. This model is useful particularly for integrated inventory systems where the vendor and the buyer form a strategic alliance for profit sharing.

The numerical examples are given to illustrate the benefit of coordination between single vendor and single buyer. Graphical representation is also presented to illustrate the proposed model. We propose an easy algorithm for determining the optimal solutions. In addition, a numerical example is presented to illustrate the proposed model. There are several extensions of this work that could constitute future research related to this field. One immediate probable extension could be to discuss the effect of shortage. Another possible extension of this work may be conducted by considering the vendor's provision of a permissible delay in payments in this integrated inventory model. Also, we can consider multi-echelon supply chains such as; single buyer-multiple vendor, multiple buyer-single vendor and multiple buyer-multiple vendor system is also proposed for the future research.

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Appendix

We want to prove the Hessian Matrix of $ITC(Q,L,\theta,n)$ at point (Q^*,L^*,θ^*) for fixed n is positive definite. We first obtain the Hessian matrix H as follows

$$H=\begin{bmatrix}\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial Q^2} & \frac{\partial^2 TRC(Q,L,n,\theta)}{\partial Q\,\partial L} & \frac{\partial^2 TRC(Q,L,n,\theta)}{\partial Q\,\partial \theta}\\ \frac{\partial^2 TRC(Q,L,n,\theta)}{\partial L\partial Q} & \frac{\partial^2 TRC(Q,L,n,\theta)}{\partial L^2} & \frac{\partial^2 TRC(Q,L,n,\theta)}{\partial L\partial \theta}\\ \frac{\partial^2 TRC(Q,L,n,\theta)}{\partial \theta\,\partial Q} & \frac{\partial^2 TRC(Q,L,n,\theta)}{\partial \theta\,\partial L} & \frac{\partial^2 TRC(Q,L,n,\theta)}{\partial \theta^2}\end{bmatrix}$$

Where

$$\frac{\partial^2 TRC(Q,L,n)}{\partial Q^2}=\frac{2D}{Q^3}\left(A+\frac{S}{n}+e^{\frac{C}{L}}\right)>0$$

$$\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial \theta^2}=\frac{iq}{\theta^2}>0$$

$$\frac{\partial^2 TRC(Q,L,n)}{\partial L^2}=\frac{D}{Q}\left(\frac{2Ce^{\frac{C}{L}}}{L^3}+\frac{C^2e^{\frac{C}{L}}}{L^4}\right)-\left(\frac{rc_b k\sigma L^{-\frac{3}{2}}}{4}\right)>0$$

$$\frac{\partial^2 TRC(Q,L,n)}{\partial Q\partial L}=\frac{\partial^2 TRC(Q,L,n)}{\partial L\partial Q}=\frac{DCe^{\frac{C}{L}}}{Q^2L^2}$$

$$\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial Q\partial \theta}=\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial \theta\,\partial Q}=\frac{snD}{2}>0$$

$$\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial L\partial \theta}=\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial \theta\,\partial L}=0$$

We proceed by evaluating the principal minor determinant of the Hessian matrix H at point $\left(Q^*, L^*, \theta^*\right)$. The first principal minor determinant of H then becomes.

$$|H_{11}| = \frac{2D}{Q^3}\left(A + \frac{S}{n} + e^{\frac{C}{L}}\right) > 0$$

$$|H_{22}| = \frac{2D}{Q^3}\left(A + \frac{S}{n} + e^{\frac{C}{L}}\right)\left(\frac{D}{Q}\left(\frac{2Ce^{\frac{C}{L}}}{L^3} + \frac{C^2 e^{\frac{C}{L}}}{L^4}\right) - \left(\frac{rc_b k\sigma L^{-\frac{3}{2}}}{4}\right)\right) - \left(\frac{DCe^{\frac{C}{L}}}{Q^2 L^2}\right)^2$$

$$= \frac{2D}{Q^3}\left(A + \frac{S}{n} + e^{\frac{C}{L}}\right)\left(\frac{2DCe^{\frac{C}{L}}}{QL^3} + \frac{DC^2 e^{\frac{C}{L}}}{QL^4} - \left(\frac{rc_b k\sigma L^{-\frac{3}{2}}}{4}\right)\right) - \left(\frac{DCe^{\frac{C}{L}}}{Q^2 L^2}\right)^2$$

$$= \frac{2D}{Q^3}\left(A + \frac{S}{n} + e^{\frac{C}{L}}\right)\left(\frac{2DCe^{\frac{C}{L}}}{QL^3} + \frac{DC^2 e^{\frac{C}{L}}}{QL^4}\right) - \frac{2D}{Q^3}\left(A + \frac{S}{n} + e^{\frac{C}{L}}\right)\left(\frac{rc_b k\sigma L^{-\frac{3}{2}}}{4}\right) - \left(\frac{DCe^{\frac{C}{L}}}{Q^2 L^2}\right)^2 > 0 \quad \text{Since}$$

$$\left(A + \frac{S}{n} + e^{\frac{C}{L}}\right)\left(\frac{4D^2Ce^{\frac{C}{L}}}{Q^4 L^3}\right) + \left(A + \frac{S}{n} + e^{\frac{C}{L}}\right)\left(\frac{2D^2C^2e^{\frac{C}{L}}}{Q^4 L^4}\right) > \left(\frac{Drc_b k\sigma L^{-\frac{3}{2}}}{2Q^3}\right)\left(A + \frac{S}{n} + e^{\frac{C}{L}}\right) + \left(\frac{DCe^{\frac{C}{L}}}{Q^2 L^2}\right)^2$$

$$\left(\frac{D}{Q^4}\right)\left(\left(A + \frac{S}{n} + e^{\frac{C}{L}}\right)\left(\frac{4DCe^{\frac{C}{L}}}{L^3}\right) + \left(A + \frac{S}{n} + e^{\frac{C}{L}}\right)\left(\frac{2DC^2e^{\frac{C}{L}}}{L^4}\right)\right) > \left(\frac{D}{Q^4}\right)\left(\left(\left(\frac{Qrc_b k\sigma L^{-\frac{3}{2}}}{2}\right)\left(A + \frac{S}{n} + e^{\frac{C}{L}}\right) + \frac{DC^2e^{\frac{C}{L}}}{L^4}\right)\right)$$

$$\left(A + \frac{S}{n} + e^{\frac{C}{L}}\right)\left(\frac{4DCe^{\frac{C}{L}}}{L^3}\right) + \left(A + \frac{S}{n} + e^{\frac{C}{L}}\right)\left(\frac{2DC^2e^{\frac{C}{L}}}{L^4}\right) > \left(A + \frac{S}{n} + e^{\frac{C}{L}}\right)\left(\frac{Qc_b k\sigma L^{-\frac{3}{2}}}{2}\right) + \left(\frac{DC^2e^{\frac{C}{L}}}{L^4}\right)$$

Therefore $|H_{22}| > 0$.

$$|H_{33}| = \frac{\partial^2 TRC(Q,L,n,\theta)}{\partial Q^2}\left(\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial L^2}.\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial \theta^2} - \left(\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial \theta \partial L}\right)^2\right)$$

$$- \frac{\partial^2 TRC(Q,L,n,\theta)}{\partial L \partial Q}\left(\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial L \partial Q}.\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial \theta^2} - \frac{\partial^2 TRC(Q,L,n,\theta)}{\partial \theta \partial L}.\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial Q \partial \theta}\right)$$

$$+\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial\theta\partial Q}\left(\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial Q\partial L}\cdot\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial L\partial\theta}-\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial L^2}\cdot\frac{\partial^2 TRC(Q,L,n,\theta)}{\partial Q\partial\theta}\right)$$

$$=\left(\frac{2D}{Q^3}\left(A+\frac{S}{n}+e^{\frac{C}{L}}\right)\right)\left(\frac{D}{Q}\left(\frac{2Ce^{\frac{C}{L}}}{L^3}+\frac{C^2e^{\frac{C}{L}}}{L^4}\right)-\left(\frac{rc_b k\sigma L^{-\frac{3}{2}}}{4}\right)\left(\frac{iq}{\theta^2}\right)-(0)^2\right)-$$

$$\left(\frac{DCe^{\frac{C}{L}}}{Q^2L^2}\right)\left(\left(\frac{DCe^{\frac{C}{L}}}{Q^2L^2}\right)\left(\frac{iq}{\theta^2}\right)-(0)\left(\frac{snD}{2}\right)\right)\quad +$$

$$\left(\frac{snD}{2}\right)\left(\left(\frac{DCe^{\frac{C}{L}}}{Q^2L^2}\right)(0)-\left(\frac{D}{Q}\left(\frac{2Ce^{\frac{C}{L}}}{L^3}+\frac{C^2e^{\frac{C}{L}}}{L^4}\right)-\left(\frac{rc_b k\sigma L^{-\frac{3}{2}}}{4}\right)\right)\left(\frac{snD}{2}\right)\right)$$

$$=\frac{2D}{Q^3}\left(A+\frac{S}{n}+e^{\frac{C}{L}}\right)\left(\left(\frac{D}{Q}\left(\frac{2Ce^{\frac{C}{L}}}{L^3}+\frac{C^2e^{\frac{C}{L}}}{L^4}\right)-\left(\frac{rc_b k\sigma L^{-\frac{3}{2}}}{4}\right)\right)\left(\frac{iq}{\theta^2}\right)\right)-\frac{DCe^{\frac{C}{L}}}{Q^2L^2}\left(\left(\frac{DCe^{\frac{C}{L}}}{Q^2L^2}\right)\left(\frac{iq}{\theta^2}\right)\right)$$

$$+\frac{snD}{2}\left(\left(-\frac{D}{Q}\left(\frac{2Ce^{\frac{C}{L}}}{L^3}+\frac{C^2e^{\frac{C}{L}}}{L^4}\right)-\left(\frac{rc_b k\sigma L^{-\frac{3}{2}}}{4}\right)\right)\left(\frac{snD}{2}\right)\right)$$

If

$$\frac{2Diq}{Q^3\theta^2}\left(A+\frac{S}{n}+e^{\frac{C}{L}}\right)\left(\frac{D}{Q}\left(\frac{2Ce^{\frac{C}{L}}}{L^3}+\frac{C^2e^{\frac{C}{L}}}{L^4}\right)-\left(\frac{rc_b k\sigma L^{-\frac{3}{2}}}{4}\right)\right)>$$

$$\frac{DCe^{\frac{C}{L}}iq}{Q^2L^2\theta^2}\left(\frac{DCe^{\frac{C}{L}}}{Q^2L^2}\right)+\left(\frac{snD}{2}\right)^2\left(\frac{D}{Q}\left(\frac{2Ce^{\frac{C}{L}}}{L^3}+\frac{C^2e^{\frac{C}{L}}}{L^4}\right)-\left(\frac{rc_b k\sigma L^{-\frac{3}{2}}}{4}\right)\right)$$

Then

$$\frac{2Diq}{Q^3\theta^2}\left(A+\frac{S}{n}+e^{\frac{C}{L}}\right)\left(\frac{D}{Q}\left(\frac{2Ce^{\frac{C}{L}}}{L^3}+\frac{C^2e^{\frac{C}{L}}}{L^4}\right)-\left(\frac{rc_bk\sigma L^{-\frac{3}{2}}}{4}\right)\right)-\frac{DCe^{\frac{C}{L}}iq}{Q^2L^2\theta^2}\left(\frac{DCe^{\frac{C}{L}}}{Q^2L^2}\right)$$

$$-\left(\frac{snD}{2}\right)^2\left(\frac{D}{Q}\left(\frac{2Ce^{\frac{C}{L}}}{L^3}+\frac{C^2e^{\frac{C}{L}}}{L^4}\right)-\left(\frac{rc_bk\sigma L^{-\frac{3}{2}}}{4}\right)\right)>0$$

Therefore $|H_{33}|>0$.

Hence for fixed n, the Hessian matrix is positive and $TRC(Q,L,n,\theta)$ is convex with respect to (Q,L,θ).