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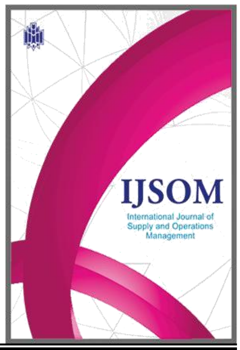

Optimal Price, Cycle Time and Advertising for an Inventory Model of Deteriorating Items: A Geometric Programming Approach

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Abstract

In recent years researchers have been interested in inventory models for deteriorating items along with determining the price of these items; given that in many real-world problems, changing the price can affect the demand level. Meanwhile, there are some other demand stimulation like advertising and discounting which can change and control the demand level of the items. In this paper, a new deteriorating inventory model was developed that considers these two demand stimulation along with pricing with multiple payments. The model has been converted and solved using geometric programming approach. Meanwhile, genetic algorithm was used as an alternative method to test the performance of GP approach. The model was implemented in real case study of food industry and numerical results and sensitivity analyses demonstrate the superiority of developed approach.

Keywords: Inventory; Deterioration; Pricing; Advertising; Multiple payments; Geometric programming.

1. Introduction

Deteriorating items are an integral part of humans' lives. Foods, fruits, vegetables, chemical materials, high-tech items, dresses, and many other goods which lose their weight, volume, freshness, efficiency, and other qualities over time are placed in the category of the deteriorating item. The values/sizes of these items alter over time and lack of attention to this issue would lead to the loss of a significant part of the related investments in their businesses. According to the relationship between demand and price, the lower price would increase the demand rate. Hence, controlling the demand is the main issue for these items. Changes in demand can affect the order quantity, the amount of deteriorated items, holding costs, capital costs, and consequently change the profit. Therefore, the recent researches have mainly focused on detecting how to control demand through demand stimulations using pricing, advertising, discounting, and so on. However, these demand stimulations are very costly and; hence, the best decision would be balancing between cost and profit of these demand stimulations. As a result, the effects of demand stimulations on an inventory model with deteriorating items is considered in this study and then the model is solved using geometric programming approach.

As an overview of the literature of deteriorating inventories, Ghare & Schrader (1963) presented an inventory model considering deteriorating items for the first time. They established the classical EOQ model with a constant rate of deterioration. Misra (1975) generalized the EOQ model to the classical EPQ model under new conditions. The pricing inventory model for deteriorating items was first proposed by Cohen(1977) where the demand was defined to be selling price dependent. Wee & Law (1999) extended the price-dependent demand model with the time value of money and allowing partial backordering. Afterwards, the model of Wee & Law (1999) was completed by defining demand function in general form and modifying the revenue terms in partial backlogging (Papachristos & Skouri, 2003). In another research, a model was developed by considering payment reduction schemes other than temporary price discounts (Arcelus et al., 2003). Also, a seller-buyer game theory model was formulated with considering trade credit policies (Abad

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& Jaggi, 2003). Thereupon, Dye et al. (2007) presented an inventory model with the aim of maximizing net present value for deteriorating items. Sana (2010) discussed a model with time-varying deterioration rate, which was determined by a probability distribution and begining after a certain time. Pervin et al., (2018) suggested an inventory model in which the deterioration rate was stochastic and demand was time-dependent. Then, Kumar Roy et al., (2020) formulated a two-warehouse inventory model considering Weibull distribution for demand function. In a similar way, the models with non-instantaneous deteriorating items have been developed (Maihami & Nakhai Kamalabadi, 2012; Soni, 2013; Li et al., 2019; Hasan et al., 2020). Likewise, multi-echelon inventory models for deteriorating items have been emerged (Maihami et al., 2016; Tiwari et al., 2016; Taghizadeh-Yazdi et al., 2020) On the other hand, many studies tried to consider factors which may decline deterioration(i.e. Huang et al., 2018; Khakzad & Gholamian, 2020; Pervin et al., 2019; Pervin et al., 2020). Recently, some new concepts have been appeared in the field of inventory management (i.e. Barman et al., 2021 and Das et al., 2021). Also in Table 1, some important researches, extracted from the literature in this area, are presented. The model is solved by geometric programming approach. The geometric programming (GP) is a type of constrained optimization which can be transformed to convex optimization and solved globally and efficiently. The model was originally developed by Duffin et al. (1967) for engineering design problems to compromise between many alternatives and so far has been used for a wide range of optimization problems. However, the first application in inventory models was performed by Kochenberger, 1971. In this way, some scholars have developed and solved inventory models using this approach. Later on, an inventory model was developed by considering quantity discount and price-dependent demand in power forms and then solved the model by reformulating into geometric programming (Lee, 1993). Then, Lee & Kim (1993) extended the previous study by incorporating advertising expenditure into the model and considering demand dependent into both selling price and advertising expenditure. This demand function was used in many other researches and other areas such as game theory models (Esmaeili & Zeephongsekul, 2010). Another extension was performed by Sadjadi & Oroujee (2005) using a combination of two previous models with some modifications. Another GP application in the inventory models goes to an inventory-production model taking into account interest and depreciation cost and fuzzy random demand (Bag et al., 2009). Likewise, Sadjadi et al. (2015) presented an improved model considering demand dependent into selling price, advertising expenditures, and customer service expenditures in presence of non-linear Interest/depreciation cost, maintenance cost, and cubic power production cost. The model is solved with 14 degrees of difficulty in GP. Unfortunately, none of the above GP-related studies have been developed for deteriorating items by considering the price and demand stimulations like advertising dependent demand, delay in payment, and demand-size-related discount simultaneously. Recently, an EOQ model with neutrosophic interval number was considered as a Posynomial GP (Mondal et al. 2020). Also, in 2020, Moradi et al. developed a price discrimination EOQ model for a manufacturer as a posynomial GP.

To put it briefly based on a review on literature, this study which arises from the existing modelling gap, considers the following contributions: I- formulating a pricing inventory model for deteriorating items with advanced payment, II- considering demand stimulations and discounts III- Converting the established model into a convex model by the GP concepts IV- solving the final model via GP solving approaches.

The rest of the paper is organized as follows: In section 2 assumptions, notations and mathematical model are introduced and discussed. Next, the model is reformulated into GP standard form and then solved using this approach. Meanwhile, in this section, a customized genetic algorithm is introduced as a benchmark of the GP approach. Numerical results and sensitivity analysis are given in section 4 to evaluate the solution method and show the model behaviour and finally conclusions are derived in Section 5.

Table 1. Literature reviews

Authors	Replenishment rate		Life time			Demand								Inflation	Discount	in payment considerations
			Decay			Deterministic							Probabilistic			
	Finite	Infinite	Variable deterioration ate	Constant deterioration ate	Constant (shelf life)	Constant	Time varying	Stock dependent	Credit-period dependent	Price sensitive	Advertisement sensitive	Quality dependent				
(Lashgari et al., 2015)		✓		✓		✓								✓		✓
(Maihami & Nakhai Kamalabadi, 2012)		✓		✓			✓			✓						
(Wu, Skouri, et al., 2014)		✓		✓				✓		✓						✓
(Chowdhury et al., 2014)		✓		✓				✓		✓						
(Srivastava & Gupta, 2014)	✓			✓			✓			✓						
(Ghiami & Williams, 2014)	✓			✓		✓										
(Taleizadeh, 2014b)		✓		✓		✓										✓
(Ghiami et al., 2013)		✓		✓	✓			✓								
(Chung et al., 2014)	✓			✓		✓										✓
(Taleizadeh & Nematollahi, 2014)		✓		✓		✓								✓		✓
(Qin et al., 2014)		✓	✓					✓		✓		✓				
(Wu, Ouyang, et al., 2014)		✓		✓	✓				✓							✓

(Sarkar et al., 2014)	✓		✓			✓										✓
(Sarkar & Saren, 2015)	✓		✓			✓										✓
(Tavakoli & Taleizadeh , 2017)		✓		✓											✓	✓
(Khan et al., 2019)		✓	✓		✓					✓	✓				✓	
(Al-Amin Khan et al., 2020)		✓	✓		✓					✓	✓					✓
This study		✓		✓						✓	✓			✓	✓	✓

2. Mathematical Modelling:

This paper stands on Lee & Kim (1993) study, with the following notation and assumption for deterministic inventory model in general form (i.e. without lead-time and shortage) as follows:

2.1. Notations

All notations which used in this paper are provided as below:

- Variables:

D: Annual demand (unit/year)
C: Purchasing cost per unit ($)
P: Selling price per unit ($)
A: Advertising cost per unit ($)
T: Cycle time duration (year)
Z: Total profit ($/year)
w_i: Dual variables
ξ:Summation of dual variables
Q: Order quantity
I(t): Inventory level at each time (t)
t_i: Time of payment of all instalments.

- Parameters:

θ: Constant deterioration rate
α: Price elasticity to demand (α>1)
β: Purchasing cost elasticity to demand (0<β<1)
γ: Advertising expenditure elasticity to demand (0<γ<1)
k: Scaling constant for demand
u: Scaling constant for purchasing cost
O: Ordering cost of inventory ($)
i: Interest rate (%)
h: Holding cost per unit ($)
λ: The length of time in which all the instalments should be paid.
ζ: The fraction of purchasing costs which is paid by pre-payments.
TH: Total holding costs ($/year)
TR: Total deterioration cost ($/year)
N: number of instalments.
TI: Total capital cost ($/year)

2.2. Assumptions

- In practice, advertising would increase the customers' intention for buying the items, and increasing in price may negatively affect their willingness to buy. To imitate this fact in this study, demand is assumed as a function of price and advertising expenses with constant elasticity according to the equation (1):

$$D = kP^{-\alpha}A^{\gamma} \qquad k > 0, \alpha > 1, \gamma > 0 \tag{1}$$

- Due to the fact that the more production rate would lead to the lesser production cost of each item, in real-world generally suppliers offer discounts which are order size-dependent. Therefore, in this paper, discount is considered based on the inverse relation of purchasing cost with demand according to the following equation:

$$C=uD^{-\beta} \qquad u>0, 0<\beta<1 \tag{2}$$

- Holding cost is a linear function of purchasing cost with a constant interest rate (i); i.e. h=iC
- The capital cost rate per unit time is assumed to be really the same as the interest rate.
- The replenishment rate is infinite and lead-time is zero.
- The number of on-hand items decreases over time due to deterioration.
- The deterioration rate (θ) is constant; in another word, a fixed fraction of inventory per unit time is corrupted.
- It is assumed that all received items are fresh and deterioration occurs as soon as the items have been received (i.e. instantaneous deterioration).
- The system produces a single type of item and no shortage and backlogging are allowed.
- The number of installments is already specified, and the size of installments is equal.

2.3. Model Development

Based on the above assumptions, the inventory level follows the pattern depicted in Fig. 1. As shown, the order quantity rises from Q' to Q to overcome the deterioration. For instance, when a fruit retailer orders for the next period, he should take into account the number of spoiled fruits during the period and hence he should order more (Q' > Q) to avoid shortages. In this figure, the inventory starts with the maximum stock level at time t=0. Then, inventory decreases during the time interval [0, T] due to demand as well as deterioration. The cycle repeats after time T. Hence, the inventory levels of the model at any time t are described by the following equation:

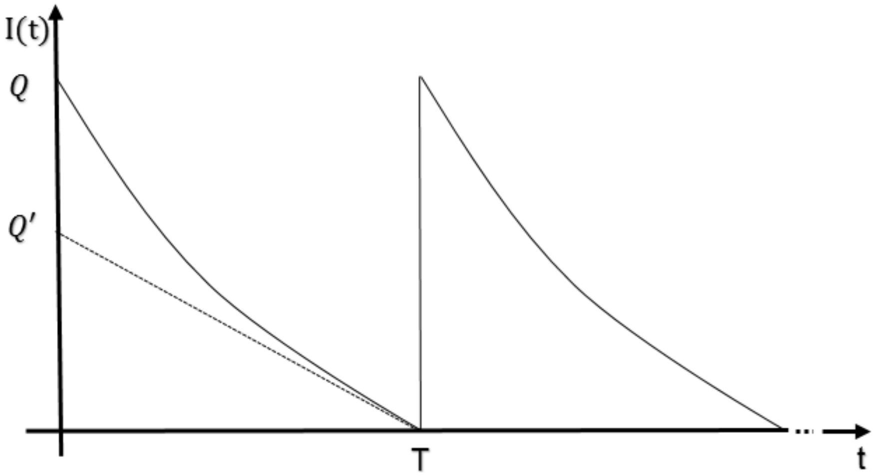

Figure 1. Deterioration and non-deterioration in the cycle time T

$$\frac{dI(t)}{dt} + \theta I(t) = -D \ ; \ \ 0 \leq t \leq T \tag{3}$$

Considering the boundary condition $I(T) = 0$ and solving equation (3) the inventory level is obtained as follows:

$$I(t) = \frac{D}{\theta}\left(e^{\theta(T-t)} - 1\right); \ \ 0 \leq t \leq T \tag{4}$$

Now, based on equation (4), the terms of the objective function can be determined as follows:

2.3.1. *Annual Holding Costs:*

Let $(\bar{I})$ denote average inventory level then by multiplying holding cost (h), the average total holding cost over the planning horizon (i.e., one year) would be:

$$\begin{aligned} TH &= \frac{h}{T}\int_0^T I(t)dt = \frac{hD}{\theta T}\int_0^T (e^{\theta(T-t)} - 1)dt = \frac{hD}{\theta^2 T}\left(e^{\theta T} - \theta T - 1\right) \\ &= \frac{i}{T\theta^2}\left(uk^{1-\beta}P^{\alpha(\beta-1)}A^{\gamma(1-\beta)}\right)\left(e^{\theta T} - \theta T - 1\right) \end{aligned} \quad (5)$$

2.3.2. Annual Deterioration Costs

To obtain the deterioration cost at each cycle, first, the amount of deteriorated items should be calculated. Using above assumptions, the number of deteriorated items is determined from the difference between order quantity and sold items. Hence, the total deterioration cost over the planning horizon would be:

$$\begin{aligned} TR &= \frac{C}{T}(Q - DT) = \frac{C}{T}\left(\frac{D}{\theta}(e^{\theta T} - 1) - DT\right) \\ &= \frac{1}{\theta T}(e^{\theta T} - 1)(uk^{1-\beta}P^{\alpha(\beta-1)}A^{\gamma(1-\beta)}) \quad - (uk^{1-\beta}P^{\alpha(\beta-1)}A^{\gamma(1-\beta)}) \end{aligned} \quad (6)$$

2.3.3. Annual Capital Costs

According to Taleizadeh (2014a), the capital cost is determined by calculating the net present value of instalments. Therefore;

$$TI = \frac{\zeta ic\lambda D(N+1)\left(e^{\theta T} - 1\right)}{2N\theta} = \frac{\zeta i\lambda(N+1)}{2N\theta}(uk^{1-\beta}P^{\alpha(\beta-1)}A^{\gamma(1-\beta)})(e^{\theta T} - 1) \quad (7)$$

2.3.4. Annual Ordering Costs

TO is equal to the cost of each order to the duration of each period.

$$TO = \frac{O}{T} \quad (8)$$

2.3.5. Total Profit

The total profit consists of selling income minus advertising, buying, ordering, holding, deteriorating, and capital costs. Advertising cost (A) is assumed per unit of demand; so, the total advertising cost would be AD. Similarly, the total revenue is calculated by multiplying the selling income of each item by the number of sold items which is represented by PD. It should be noted that since the shortage is not allowed the total sold items would be the same annual demand. Hence, the inventory model can be presented as follows:

$$\text{Max } f(P, A, T, C, D) = PD - AD - CD - \frac{O}{T} - H - R - I \quad (9)$$

$$P, A, T, C, D \geq 0 \quad (10)$$

It should be noted that D and C are indirect variables of the model; since they are defined based on other variables of the model (i.e. P, M) as shown in equations (1) and (2).

By replacing equations (5)-(7) into (9)-(10) and also by replacing indirect variables with equations (1)-(2) the following optimization model would be obtained:

$$\begin{aligned} \text{Max } f(P, A, T) = {} & (P - A)(kP^{-\alpha}A^{\gamma}) - \frac{O}{T} \\ & - \frac{i}{T\theta^2}\left(uk^{1-\beta}P^{\alpha(\beta-1)}A^{\gamma(1-\beta)}\right)\left(e^{\theta T} - \theta T - 1\right) \\ & - \frac{\zeta i\lambda(N+1)T + 2N}{2N\theta T}(uk^{1-\beta}P^{\alpha(\beta-1)}A^{\gamma(1-\beta)})(e^{\theta T} - 1) \end{aligned} \quad (11)$$

$$P, A, T \geq 0$$

As can be seen, the resulted model in the form of (11) is an unconstrained nonlinear problem. The optimal solution of such models can be obtained using the geometric programming (GP) approach as brought and described in the next section.

3. Solution Approach

This section will investigate two solution approaches. First, the GP approach is briefly introduced and then the model is reformulated and solved using this approach. Afterward, a meta-heuristic approach is developed as a benchmark to check the performance of the GP approach.

The geometric programming is a type of non-linear optimization model with two important characterizations (Duffin et al., 1967):

1- Ability to solve problems even in large scales effectively and with full reliability.
2- Ability to perform calculations at a fraction of time.

Two basic concepts of the GP are monomial and posynomial functions which are defined below:

Definition 1. Monomials and posynomials (Boyd & Vandenberghe, 2004)
A function h: $R^n \rightarrow R$ with $h = \{y|y > 0\}$, which is defined as

$$h(y) = ky_1^{a_1}y_2^{a_2} \dots y_n^{a_n}, \tag{12}$$

where $k \geq 0, y_i > 0$ and $a_i \in R$, is called a *monomial function*, and the sum of K monomial is called a posynomial function (with K terms), as below
where $k_j \geq 0, y_i > 0$,

$$h(y) = \sum_{j=1}^{J} k_j y_1^{a_{1j}} y_2^{a_{2j}} \dots y_n^{a_{nj}}, \tag{13}$$

Definition 2. The standard form of GP (Boyd & Vandenberghe, 2004)
A geometric program is an optimization problem of the form

Minimize f(**y**)
Subject to: $h_i(\mathbf{y}) \leq 1$ $i = 1,\dots, m$ (14)
$g_j(\mathbf{y}) = 1$ $j = 1,.., n$

Where h_i are posynomial and g_j are monomial and **y** is a vector of variables.
Based on the above definitions, to make a GP form, equation (9) should be reformulated into a minimization problem with the posynomial form. So, a new variable (Z > 0) was defined as follows:

$$\mathrm{PD} - \mathrm{AD} - \mathrm{CD} - \frac{\mathrm{O}}{\mathrm{T}} - \mathrm{H} - \mathrm{R} - \mathrm{I} = \mathrm{Z} \tag{15}$$

Equation (8) needs to be positive because it is a type of revenue otherwise the inventory system is not cost-effective. The standard GP is in the minimization form, and also by replacing (15) into (9) the unconstrained optimization model can be converted into the following constrained optimization model.

$$\mathrm{Min}\ \mathrm{Z}^{-1} \tag{16}$$

S.t

$$\mathrm{PD} - \mathrm{AD} - \mathrm{CD} - \frac{\mathrm{O}}{\mathrm{T}} - \mathrm{H} - \mathrm{R} - \mathrm{I} \geq \mathrm{Z} \tag{17}$$

$$\mathrm{Z}, \mathrm{P}, \mathrm{A}, \mathrm{T} \geq 0$$

Equation (17) is considered in the form of inequality to create an upper bound on Z and to prevent the tending into infinity. The new optimization model (16)-(17) is not yet in accordance with the standard form of GP. As described in definition 1, in Posynomial form all coefficients should be positive. In addition, the constraint is not coincident with definition 2. However, dividing Eq. (17) into PD will solve both problems:

$$\mathrm{P}^{-1}\mathrm{D}^{-1}\mathrm{Z} + \mathrm{P}^{-1}\mathrm{C} + \mathrm{P}^{-1}\mathrm{A} + \mathrm{O}\mathrm{T}^{-1}\mathrm{P}^{-1}\mathrm{D}^{-1} + \mathrm{H}\mathrm{P}^{-1}\mathrm{D}^{-1} + \mathrm{R}\mathrm{P}^{-1}\mathrm{D}^{-1} + \mathrm{I}\mathrm{P}^{-1}\mathrm{D}^{-1} \leq 1 \tag{18}$$

Again, replacing C, D, H and F into Eq.(20), yields:

$$k^{-1}P^{\alpha-1}A^{-\gamma}Z + P^{-1}A + uk^{-\beta}P^{\alpha\beta-1}A^{-\gamma\beta} + k^{-1}OP^{\alpha-1}A^{-\gamma}T^{-1} + \frac{i}{\theta^2 T}\left(uk^{-\beta}P^{\alpha\beta-1}A^{-\gamma\beta}\right)\left(e^{\theta T} - \theta T - 1\right) + \frac{\zeta i\lambda(N+1)T + 2N}{2N\theta T}\left(uk^{-\beta}P^{\alpha\beta-1}A^{-\gamma\beta}\right)\left(e^{\theta T} - 1\right) \leq 1 \quad (19)$$

However, equation (19) is not still in Posynomial form; because of negative phrases in the last term. This problem can be solved using the Taylor series. To reach the minimum residual, the fourth-order approximation of exponential Taylor series was used:

$$e^{\theta T} \approx 1 + \theta T + \frac{\theta^2 T^2}{2} + \frac{\theta^3 T^3}{6} + \frac{\theta^4 T^4}{24} \quad \mathbf{(20)}$$

With this replacement, the negative phrases will be discarded. As mentioned previously, this standard GP model can achieve the global optimal solution in a short time. Unlike traditional non-linear programming solution approaches, there is no need to check the convexity and globally of solutions regarding the following lemma.
Therefore, the model can be represented in standard form as below:

$$\text{Min } Z^{-1} \quad (21)$$

S.t

$$\begin{aligned} &k^{-1}P^{\alpha-1}A^{-\gamma}Z + P^{-1}A + uk^{-\beta}P^{\alpha\beta-1}A^{-\gamma\beta} + k^{-1}OP^{\alpha-1}A^{-\gamma}T^{-1} + \frac{iT}{2}\left(uk^{-\beta}P^{\alpha\beta-1}A^{-\gamma\beta}\right) \\ &+ \frac{i\theta T^2}{6}\left(uk^{-\beta}P^{\alpha\beta-1}A^{-\gamma\beta}\right) + \frac{i\theta^2 T^3}{24}\left(uk^{-\beta}P^{\alpha\beta-1}A^{-\gamma\beta}\right) + uk^{-\beta}P^{\alpha\beta-1}A^{-\gamma\beta} \\ &+ \frac{\theta T}{2}\left(uk^{-\beta}P^{\alpha\beta-1}A^{-\gamma\beta}\right) + \frac{\theta^2 T^2}{6}\left(uk^{-\beta}P^{\alpha\beta-1}A^{-\gamma\beta}\right) + \frac{\theta^3 T^3}{24}\left(uk^{-\beta}P^{\alpha\beta-1}A^{-\gamma\beta}\right) \\ &+ \frac{\zeta i\lambda(N+1)T}{2N}\left(uk^{-\beta}P^{\alpha\beta-1}A^{-\gamma\beta}\right) + \frac{\zeta i\lambda(N+1)\theta T^2}{4N}\left(uk^{-\beta}P^{\alpha\beta-1}A^{-\gamma\beta}\right) \\ &+ \frac{\zeta i\lambda(N+1)\theta^2 T^3}{12N\theta}\left(uk^{-\beta}P^{\alpha\beta-1}A^{-\gamma\beta}\right) + \frac{\zeta i\lambda(N+1)\theta^3 T^4}{48N}\left(uk^{-\beta}P^{\alpha\beta-1}A^{-\gamma\beta}\right) \leq 1 \end{aligned} \quad (22)$$

$$Z, P, A, T > 0 \quad (23)$$

Lemma 1. If a model is in Posynomial/Monomial GP standard form or is transformable to Posynomial GP standard form, the model can be converted into a convex optimization model and hence solved with GP (Duffin et al., 1967).

Proof: Appendix. A

As shown the model (i.e. Equations (21), (22), and (23)), is quite compatible with GP standard form, and hence according to lemma 1, the model could be convex and have a global optimal solution.
It should be noted that the model has 15 (number of terms-(number of variables + 1)) degrees of difficulty and so cannot be solved using commercial solvers such as CPLEX; so, the model was coded and solved using MATLAB software.

To reveal the validation of the developed approach, the results should be compared with an approach with validity in terms of performance. Meta-heuristics can be used successfully in this way; since they produce lower bounds near to optimal solutions which can be easily compared with the global optimal solution determined by GP. Especially, in this research, a customized genetic algorithm is introduced which is designed for solving multivariate unconstrained non-linear programming problems. The algorithm was basically developed by Ghiami and Williams (2014), which has been successfully used in inventory models with deteriorating items. The stepwise procedure of the algorithm is shown as follows:

1. Specify the GA parameters as follows:
    a. Chromosome representation: by a three-dimensional vector X= (P, A, T) where all three variables are real positive numbers.
    b. Initial population: which is randomly generated in size of 1000.
    c. Genetic operations: consisting of 80% one-point cross-over operations and 20% mutation based on Gaussian distribution function with zero mean.
    d. Fitness function: based on the value of the objective function (10)-(11) for each solution.
2. Generate the first population.
3. Choose the best solutions as elite children to survive. In this study, the first 100 children (with the highest objective values) were selected as elite solutions.
4. If at least one of the following three stopping criteria (i.e. Maximum iteration=1500, function tolerance= 1×10^{-5} , stall generation=100) is reached, go to 7.

5. Generate the new children from the previous population (parents) by mutation and crossover operations to complete the population.
6. Find the best solution in the population based on optimal fitness function values and go to 4.
7. Save the optimal solution.

4. Numerical Results and Evaluations

In this study, a real inventory system of the food industry supply chain is investigated. It should be noted that due to some restrictions on access to the actual data, some data are estimated with approximations close to reality. Meanwhile, there are some restrictions in the relationship between parameters which are oriented from the nature of the model. Theorem (1) describes these relationships:

Theorem 1. The first order feasibility conditions for the model holds when the following conditions are met,

$$\gamma < \alpha - 1, \gamma\beta > \alpha\beta - 1 \text{ and } \beta < \frac{1}{\alpha-\gamma} \tag{24}$$

Proof: Appendix. B

The case study was solved on a computer with an Intel Core i7 (2.2 GHz) CPU and 6.00 GB of RAM. Both GP and GA approaches were solved using MATLAB 2017. The results have been reported below. Meanwhile, a sensitivity analysis on the parameters of the model was performed to check the consistency of the model by demonstrating the changing effects of these parameters on optimal solution and decision variables.

a. Case study:

To illustrate the performance of the model, as a case study, a bakery product retailer (including biscuit, cake, cookie, cracker, pie, strudel, tart, wafer and so on) in Tehran has been considered. The shop manager wants to control the demand by setting the optimal price and advertising values. Currently, in the market the elasticity parameters are: $k = 2 \times 10^6$, $\alpha = 2.1$, $\beta = 0.01$, $\gamma = 0.03$. The manager knows based on equation (2) with $u = 50$, the production costs and also raw material costs would decrease, if demand rate increased. Formal inflation rate is 5% and deterioration rate is 11%. Setup costs is around 100 TM*. The supplier asks the retailer to pay to fraction of the purchasing cost as three equal instalments (ζ=0.4) within 126 days (λ=0.35) at the time the order is placed. The manager has to calculate the optimal price, advertising cost and cycle time in order to stay competitive in the market. Before running the model, the elasticity parameters were checked to satisfy theorem 1. The detailed results of the model solution are shown in Table 2. As observed, while the GP is dealing with the model which uses the Taylor series as an approximation (21)-(22)-(23) and GA is dealing with the original model (11); the GP and GA results are quite similar, which confirms the validity of the global optimal solution of GP approach. In any case, GP was extremely faster than GA.

Table2: Case study results

method	Total profit	Variables			Income	Costs						Elapsed time (s)
		P	A	T	Selling	Purchasing	Advertising	TO	TH	TR	TI	
GP	6051.6	95.7	1.3	0.41	13358	6638.1	190.8	240.7	69.9	153.9	13.1	0.3
GA	6051.6	95.7	1.3	0.41	13358	6638.1	190.8	240.7	69.9	153.9	13.1	110.5

b. Sensitivity Analysis:

This section shows the effects of the elasticity changes on the decision variables and the total profit by 25% and 50% increasing or decreasing the elasticity parameters in both GP and GA approaches. The results are brought in Tables 3, 4, and 5, and the findings are summarized as follows:

α : This parameter represents the price elasticity. As shown in Table 3, the objective function and also income increase by decreasing the α and vice versa (Figs: 2, 3). Instead, cycle time (T*) is in line with α in rise and fall and also, the changes in α affect T* exponentially (Fig 4). As a reason, the decline in price elasticity has a profitable effect on demand

* Thousand Tomans

which in turn will lead to growth in holding, ordering, purchasing, advertising, deteriorating costs, and reduce profit margin. Consequently, cycle time will increase to neutralize the reduction effect of total profit in the optimal solution. On top of that, this parameter has interesting effects on purchasing, holding, deterioration, and capital costs as shown in Fig.5. Fig.5 illustrates that the change in α leads to a concave curve of changing total ordering cost. Besides, as shown in Table 3 the GP shows its advantage over GA at lower α values. α is the most influential elasticity parameter such that the changes in α would drastically affect cost and income terms and consequently, objective function changes considerably by changes in α.

β: This parameter represents the purchasing cost elasticity. As shown in Table 4, all three decision variables have a reciprocal relationship with β. Instead, the objective function, income and all types of costs are in line with β in rise and fall. Obviously, the decline in β will only increase the purchasing cost per unit (C) that consequently reduce the total profit (see Fig 6). Hence, the decision variables increase in an attempt to compensate for this reduction in the optimal solution.

γ: This parameter represents the advertising cost elasticity. As shown in Table 5, decision variables and advertising costs are in line with this parameter, but other costs are in the opposite direction of this parameter. Since the growth in advertising cost elasticity has a profitable effect on demand which in turn will lead to a decline in holding, ordering, purchasing, and deteriorating costs. Similar to α, γ has also interesting effects on the objective function, income, purchasing cost (per item), and cycle time, as shown in Figs:7-10. However, the changes in this parameter are not much, comparing the changes with α. In short, all cost and income terms are almost stable with changes in γ. In addition, based on Figs 7-10, both objective function (F*) and income term, are a convex function of γ and T* and in contrast, C* is concave with respect to γ .

Table 3. Sensitive analysis of new model for α

Changes (%)	method	Total profit	variables			Income	Costs					
			P	A	T	Selling	Purchasing	Advertising	TO	TH	TR	TI
+50%	GP	4.63	108.1	1.02	4.60	84.80	39.30	0.80	21.70	5.39	11.87	1.10
	GA	4.60	108.0	1.02	4.58	84.97	39.42	0.80	21.79	5.38	11.85	1.09
+25%	GP	438.77	87.5	1	1.17	1396.8	776.18	15.96	84.76	23.91	52.61	4.56
	GA	438.76	87.5	1	1.17	1396.8	776.19	15.96	84.77	23.91	52.60	4.56
0	GP	6051.6	95.7	1.3	0.41	13358	6638.1	190.83	240.78	69.98	153.95	13.16
	GA	6051.6	95.7	1.3	0.41	13358	6638.1	190.83	240.78	69.98	153.95	13.16
-25%	GP	76628	135.5	2.5	0.16	122310	42153	2329.7	601.32	176.32	387.91	33.01
	GA	76628	135.5	2.5	0.16	122310	42154	2329.8	602.40	176	387.21	32.95
-50%	GP	1463500	2478.5	70.8	0.20	1537500	29086	43929	500	146.5	322.30	27.44
	GA	1409000	538.7	16.5	0.03	1588600	136120	48717	2801.1	121.64	267.62	22.72

Table 4. Sensitive analysis of new model for β

Changes (%)	method	Total profit	variables			Income	Costs					
			P	A	T	Selling	Purchasing	Advertising	TO	TH	TR	TI
+50%	GP	6225.8	92.8	1.32	0.40	13815	6904.4	197.35	245.51	71.36	157.00	13.42
	GA	6225.8	92.8	1.32	0.40	13815	6904.4	197.35	245.51	71.36	157.00	13.42
+25%	GP	6137.6	94.2	1.34	0.41	13583	6769.3	194.05	243.13	70.66	155.46	13.29
	GA	6137.6	94.2	1.34	0.41	13583	6769.3	194.05	243.13	70.66	155.46	13.29
0	GP	6051.6	95.7	1.36	0.41	13358	6638.1	190.83	240.78	69.98	153.95	13.16
	GA	6051.6	95.7	1.36	0.41	13358	6638.1	190.83	240.78	69.98	153.95	13.16
-25%	GP	5967.8	97.2	1.38	0.41	13140	6510.7	187.71	238.54	69.29	152.44	13.03
	GA	5967.8	97.2	1.38	0.41	13140	6510.7	187.71	238.54	69.29	152.44	13.03
-50%	GP	5886.1	98.7	1.41	0.42	12926	6386.9	184.65	236.29	68.63	150.99	12.91
	GA	5886.1	98.7	1.41	0.42	12926	6386.9	184.65	236.29	68.63	150.99	12.91

Table 5. Sensitive analysis of new model for γ

Changes (%)	method	Total profit	variables			Income	Costs					
			P	A	T	Selling	Purchasing	Advertising	TO	TH	TR	TI
+50%	GP	6103.0	97.15	2.08	0.41	13464	6596.0	288.51	240.03	69.75	153.47	13.12
	GA	6103.0	97.15	2.08	0.41	13464	6596.0	288.51	240.03	69.75	153.47	13.12
+25%	GP	6072.3	96.45	1.72	0.41	13400	6611.8	239.29	240.32	69.84	153.64	13.13
	GA	6072.3	96.45	1.72	0.41	13400	6611.8	239.29	240.32	69.84	153.64	13.13
0	GP	6051.6	95.77	1.36	0.41	13358	6638.1	190.83	240.78	69.98	153.95	13.16
	GA	6051.6	95.77	1.36	0.41	13358	6638.1	190.83	240.78	69.98	153.95	13.16
-25%	GP	6043.3	95.08	1.01	0.41	13343	6677.8	142.96	241.54	70.17	154.38	13.19
	GA	6043.3	95.08	1.01	0.41	13343	6677.8	142.96	241.54	70.17	154.38	13.19
-50%	GP	6051.6	94.40	0.67	0.41	13364	6735.8	95.46	242.54	70.49	155.08	13.25
	GA	6051.6	94.40	0.67	0.41	13364	6735.8	95.46	242.54	70.49	155.08	13.25

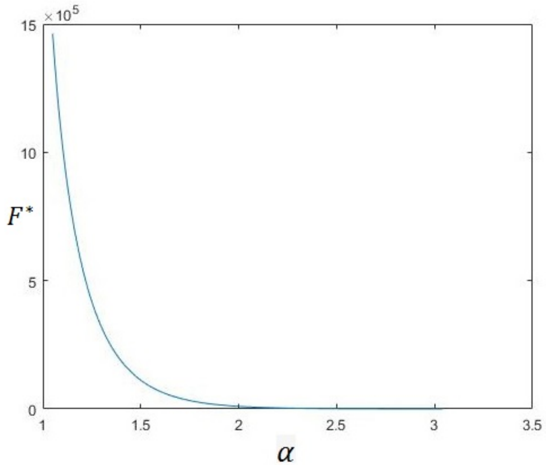

Figure 2. The changes of α versus optimal objective (F*)

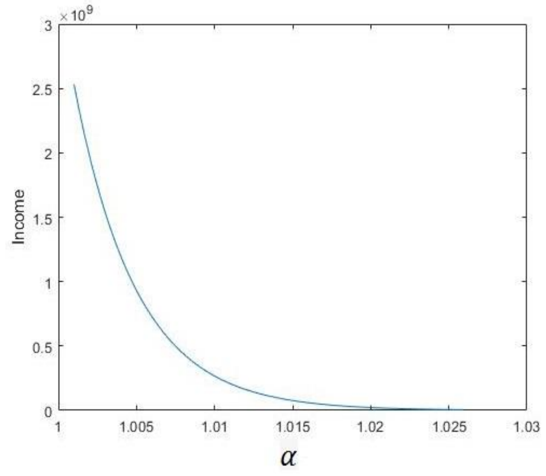

Figure 3. The changes of α versus income

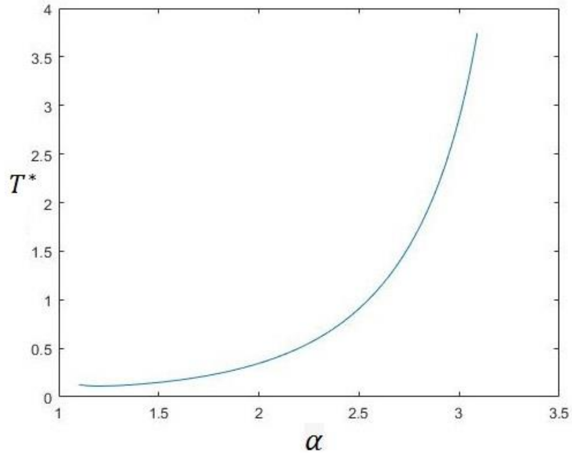

Figure 4. The changes of α versus optimal cycle time (T*)

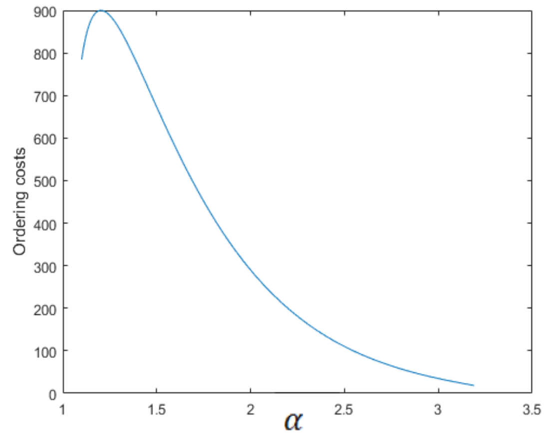

Figure 5. The changes of α versus annual ordering costs

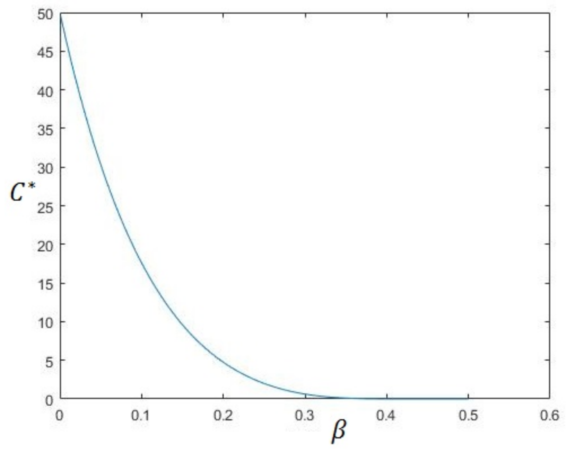

Figure 6. The changes of β versus optimal item cost (C*)

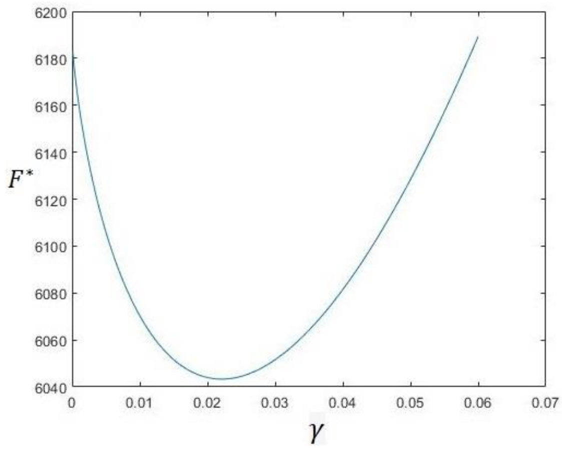

Figure 7. The changes of γ versus optimal objective (F*)

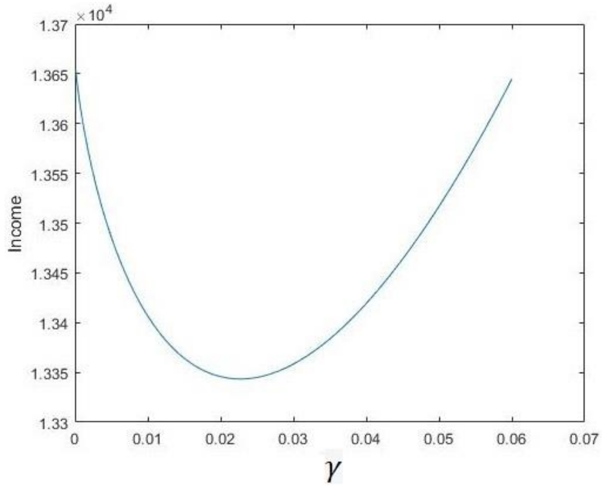

Figure 8. The changes of γ versus income

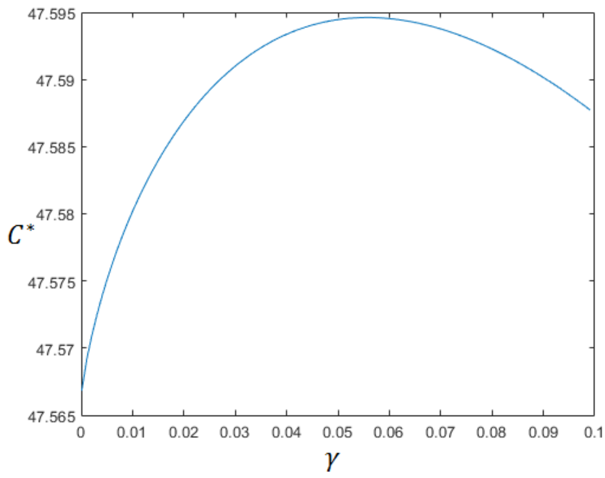

Figure 9. The Changes of γ versus optimal item cost (C*)

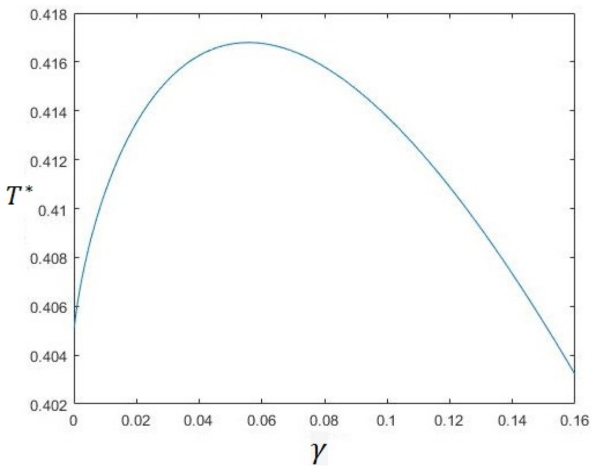

Figure 10. The changes of γ versus optimal cycle time (T*)

5. Conclusion

This study has contributed to defining an inventory model for deterioration items with consideration of some demand stimulations. In other words, the demand function is defined in response to the selling price and advertising cost. In addition, instalment payment is considered to make the model closer to reality. Accordingly, some new terms including advertising cost and capital cost would be added to the model. Meanwhile. It is assumed that production cost is discounted with increasing demand. As a result, the objective function consists of selling income in one hand and buying, ordering, advertising, holding, deteriorating and capital costs in another hand. All terms of the objective function are defined based on the basis of considering deteriorating conditions. The mathematical model is derived to simultaneously determine the optimal selling price, advertising cost, and cycle time.

The model is then reformulated into the standard form of geometric programming and solved using GP approach. To evaluate the performance of GP approach, the model is also solved using a customized genetic algorithm and the results were used as a benchmark to check the globality of GP optimal solution. A real-world numerical example from the food industry (i.e. bakery products) is given to illustrate the performance of the model. Furthermore, sensitivity analysis is carried out with respect to the key parameters (i.e. elasticity parameters), and some managerial insights are introduced.
As practical implementation of this study, it should be mentioned that the values of elasticity parameters depend on market and could not be determined by the business manager. The results show that fluctuation in α would have a significant effect on all main variables (i.e. P, T, A) of the model such that increasing in α will highly increase optimal price (P) with a significant increasing effect in length of cycle time (T), and decreasing in α will highly decrease advertising cost (A). While fluctuation in γ would not affect the business seriously. Also, β has direct relation with profit function and reverse relation with optimal price. Therefore, it can be concluded that this industry is quite sensitive to pricing and proper pricing is the first issue of this industry. After that, more attention should be paid on considering right investment cost and right

length of cycle time. This is the topic that this optimization model can help industry achieve to reach into the optimal results.

For future research, the model can be extended by relaxing the assumptions of this research; for example, by allowing the shortage, using non-zero lead-time, using instantaneous/non-instantaneous deterioration rate functions, etc. Also, the model can be extended to multi-echelon inventory systems by considering the inventory optimization of all layers instantaneously. Finally, the model can be developed in uncertain conditions using fuzzy or stochastic parameters.

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Appendix A

The proof is straightforward from the idea originally developed by Boyd & Vandenberghe (2004).
Consider an optimization model as follows,

$$\text{Minimize} \quad h_0(y) \tag{A.1}$$
$$\text{Subject to} \quad h_i(y) \leq 1 \;,\; i=1,\ldots,n \tag{A.2}$$
$$g_i(y) = 1,\; i=1,\ldots,m \tag{A.3}$$
$$y > 0 \tag{A.4}$$

Where h_j are posynomial and g_j are monomials.

Suppose that the variables are redefined as $v_i = \log y_i$,$c = \log k$,$f_i = \log g_i$ and therefore $y_i = e^{v_i}$ and $g_i = e^{f_i}$. Now, let h(y), like Equation (A.2), be a monomial function of y; then we have:

$$\begin{aligned} \acute{h}(v) &= h(y) \\ &= h(e^{v_1}, \ldots, e^{v_n}) \\ &= k(e^{v_1})^{a_1} \ldots (e^{v_n})^{a_1} \\ &= e^{a^T v + c} \end{aligned} \tag{A.5}$$

Likewise, let h(y) be posynomial function like Equation (A.5), then the model (A.1) to (A.4) is transformed to following form,

$$\text{Minimize} \sum_{j=1}^{J_0} e^{a_{0j}^T v + c_{0j}} \tag{A.6}$$
$$\text{Subject to} \sum_{j=1}^{J_i} e^{a_{ij}^T v + c_{ij}} \leq 1 \;,\; i = 1, \ldots, n \tag{A.7}$$
$$e^{f_i^T v + g_i} = 1,\; i = 1, \ldots, m \tag{A.8}$$
$$v > 0 \tag{A.9}$$

where $a_{ij} \in R^2$ and $i = 0, \ldots, m$.

Finally, by taking the logarithm of above model (in both objective function and constraints) the final model will be resulted as shown in (A.10)-(A.12):

$$\text{Minimize} \quad \log \sum_{j=1}^{J_0} e^{a_{0j}^T v + c_{0j}} \tag{A.10}$$
$$\text{Subject to} \quad \acute{h}_i(v) = \log \sum_{j=1}^{J_i} e^{a_{ij}^T v + c_{ij}} \leq 0 \;,\; i=1,\ldots,n \tag{A.11}$$
$$\acute{g}_i(v) = f_i^T v + g_i = 0,\; i=1,\ldots,m \tag{A.12}$$
$$v > 0$$

Since $\acute{h}_i$ is convex and $\acute{g}_i$ are affine, the model is convex, which completes the proof. ■

Appendix B

Let solve the model using the procedure explained by Duffin et al. (1967) based on duality theorem. The corresponding GP (i.e. equations (21)-(22)-(23)) dual model would be stated as follows:

$$\begin{aligned} \text{(DP)} \max g(w) = & \left[\frac{1}{w_0}\right]^{w_0} \left[k^{-1}\frac{\xi}{w_1}\right]^{w_1} \left[\frac{\xi}{w_2}\right]^{w_2} \left[uk^{-\beta}\frac{\xi}{w_3}\right]^{w_3} \left[k^{-1}O\frac{\xi}{w_4}\right]^{w_4} \\ & \left[\frac{1}{2}\text{iuk}^{-\beta}\frac{\xi}{w_5}\right]^{w_5} \left[\frac{1}{6}\text{iuk}^{-\beta}\theta\frac{\xi}{w_6}\right]^{w_6} \left[\frac{1}{24}\text{iuk}^{-\beta}\theta^2\frac{\xi}{w_7}\right]^{w_7} \left[\text{uk}^{-\beta}\frac{\xi}{w_8}\right]^{w_8} \\ & \left[\frac{1}{2}\text{uk}^{-\beta}\theta\frac{\xi}{w_9}\right]^{w_9} \left[\frac{1}{6}\text{iuk}^{-\beta}\theta^2\frac{\xi}{w_{10}}\right]^{w_{10}} \left[\frac{1}{24}\text{iuk}^{-\beta}\theta^3\frac{\xi}{w_{11}}\right]^{w_{11}} \left[\frac{\zeta i\lambda(N+1)}{2N}\text{uk}^{-\beta}\frac{\xi}{w_{12}}\right]^{w_{12}} \\ & \left[\frac{\zeta i\lambda(N+1)\theta}{4N}\text{uk}^{-\beta}\frac{\xi}{w_{13}}\right]^{w_{13}} \left[\frac{\zeta i\lambda(N+1)\theta^2}{12N}\text{uk}^{-\beta}\frac{\xi}{w_{14}}\right]^{w_{14}} \left[\frac{\zeta i\lambda(N+1)\theta^3}{48N}\text{uk}^{-\beta}\frac{\xi}{w_{15}}\right]^{w_{15}} \end{aligned} \tag{B.1}$$

S.t

$$w_0 = 1 \tag{B.2}$$
$$-w_0 + w_1 = 0 \tag{B.3}$$
$$(\alpha - 1)(w_1 + w_4) + (\alpha\beta - 1)\begin{pmatrix} w_3 + w_5 + w_6 + w_7 + w_8 + w_9 + w_{10} \\ +w_{11} + w_{12} + w_{13} + w_{14} + w_{15} \end{pmatrix} = w_2 \tag{B.4}$$
$$\gamma(w_1 + w_4) + \gamma\beta\begin{pmatrix} w_3 + w_5 + w_6 + w_7 + w_8 + w_9 + w_{10} \\ +w_{11} + w_{12} + w_{13} + w_{14} + w_{15} \end{pmatrix} = w_2 \tag{B.5}$$
$$w_5 + 2w_6 + 3w_7 + w_9 + 2w_{10} + 3w_{11} + w_{12} + 2w_{13} + 3w_{14} + 4w_{15} = w_4 \tag{B.6}$$
$$\boldsymbol{\xi} = \sum_{i=1}^{15} \boldsymbol{w_i} \tag{B.7}$$

Where w_i are dual variables.
By combining equations (B.4) and (B.5), equation (B.8) is achieved in the following. Now, by substituting (B.8) in (B.4) and (B.5), equations (B.9) and (B.10) are respectively obtained as follows

$$w_3 + w_5 + w_6 + w_7 + w_8 + w_9 + w_{10} + w_{11} + w_{12} + w_{13} + w_{14} + w_{15} = \frac{(\gamma - \alpha + 1)(1 + w_4)}{\alpha\beta - \gamma\beta - 1} \quad \text{(B.8)}$$

$$w_2 = (1 + w_4)((\alpha - 1) + \frac{(\alpha\beta - 1)(\gamma - \alpha + 1)}{\alpha\beta - \gamma\beta - 1}) \quad \text{(B.9)}$$

$$w_2 = (1 + w_4)(\gamma + \frac{\gamma\beta(\gamma - \alpha + 1)}{\alpha\beta - \gamma\beta - 1}) \quad \text{(B.10)}$$

Obviously, these two equations (i.e.(B.9) and (B.10)) are not necessarily true *for all values* of α, β and γ. ; So, the terms should be compared one by one. These comparisons can lead to the following three scenarios:

i. *First scenario:*

$$\gamma = \alpha - 1, \qquad \gamma\beta = \alpha\beta - 1 \quad \text{(B.11)}$$

The first scenario is the simplest ones with considering the equal relations. However, by inserting the first term into the second term, we will have:

$$\gamma\beta = \beta(\alpha - 1) \neq \alpha\beta - 1 \quad \text{(B.12)}$$

Which is an impossible relation and so, the first scenario is rejected and set aside.

ii. *Second scenario:*

$$\gamma > \alpha - 1, \qquad \gamma\beta < \alpha\beta - 1 \quad \text{(B.13)}$$

In this scenario, the first term was set to "greater than" and second term was set to "lesser than". Again, like previous scenario, by inserting the first term into the second term, the following relations will be obtained;

$$\gamma\beta > \beta(\alpha - 1), \qquad \gamma\beta < \alpha\beta - 1 \quad \text{(B.14)}$$

Which is impossible again; since β is defined in the range [0, 1] and so the second scenario is also rejected and set aside.

iii. *Third scenario:*

$$\gamma < \alpha - 1, \qquad \gamma\beta > \alpha\beta - 1 \quad \text{(B.15)}$$

In this scenario, the first term was set to "lesser than" and second term was set to "greater than" against the previous scenarios. Now, inserting the first term into the second term gives the following results;

$$\gamma\beta < \beta(\alpha - 1), \qquad \gamma\beta > \alpha\beta - 1 \quad \text{(B.16)}$$

From equation (B.16), it follows that:

$$\alpha\beta - 1 < \gamma\beta < \beta(\alpha - 1) \quad \text{(B.17)}$$

and by dividing Eq. (B.17) into β, we derive:

$$\frac{\alpha\beta - 1}{\beta} < \gamma < \alpha - 1 \quad \text{(B.18)}$$

Which is acceptable and valid relation in the ranges of these three parameters.
But however, $\gamma < \alpha - 1$ generates a negative value in equation (B.8); whilst all the dual variables (w_i) should be positive, so the denominator in equation (B.8) should be also set to be negative, therefore

$$\alpha\beta - \gamma\beta - 1 < 0 \Rightarrow \beta < \frac{1}{\alpha - \gamma} \quad \text{(B.19)}$$