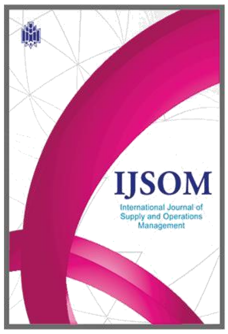
International Journal of Supply and Operations Management



IJSOM

November 2020, Volume 7, Issue 4, pp. 310-321
ISSN-Print: 2383-1359
ISSN-Online: 2383-2525
www.ijsom.com

Analyzing Impatience in Multiserver Markovian Queues

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Abstract

Reneging and Balking are two facts of customer impatience. In traditional queuing literature, customer impatience was not often considered. However, for the last few decades, queuing theorists have been trying to integrate aspect of balking and reneging into modeling of queues in a holistic way. This paper is an extension of the work in the same direction. We consider a multiserver Markovian queuing model under the assumption that customers are state aware so that their impatience is state dependent. We derive the generating function of the stationary system size distribution and obtain mean system size along with other performance measures.

Keywords: Balking; Impatience; Queuing; Reneging; Performance measures; Multiserver queue.

1. Introduction

Queues are a very common day phenomenon which is present almost. Because of constraints with service delivery mechanism, queues or waiting lines arise when the customers does not get service immediately i.e. when the demand for a service facility goes beyond the capacity of that facility. Waiting is disliked as customers are hard pressed for time. The act of having to wait in a line or queue often induces impatience in arriving customers. In queuing systems, such impatience can be of two types – balking and reneging. If an arriving customer finds the service facility to be non-empty and leaves without joining the queue, then such a phenomenon is called balking. Haight (1957) was possibly the first to introduce the concept of balking.

Reneging is the other commonly observed impatient customer behavior. It occurs when arriving customers join the queuing system, get impatient and leave before completion of service. Barrer (1957) has outlined two types of reneging - viz. reneging till beginning of service (R_BOS) and reneging till end of service (R_EOS). Additionally, depending on the nature of the reneging rate, queuing literature provides for splitting reneging into two types viz. position independent reneging (PIR) and position dependent reneging (PDR). Choudhury and Medhi (2011B) have illustrated these two types of reneging. Similar to reneging, the balking can also be classified into two parts viz. state independent balking (SIB) and state dependent balking (SDB). (Choudhury and Medhi, 2012).

In this paper, we analyze a multiserver Markovian queuing system under the assumtion that customers may balk as well as renege. We consider a specific balking rule where the balking probability of the customer decreases as the state of the system goes up i.e. balking is state dependent (SDB). We also assume position dependent reneging (PDR) which is very relevant from practical point of view. Here the reneging rate is a function of the position of the customer. For example, in case of life insurance business where the purchase of a policy refers to the arrival of a customer into the queuing system (insurance firm), the processed application can be called the departure from the queuing system, the claim processing department is considered as a server and the system capacity (the number of policies it can accommodate) is taken as infinite. The claims are processed in order of their arrival (i.e. the queue discipline is FCFS). Jain et al., (2014) have assumed that the probability of joining the firm (i.e. chances of purchasing a policy) is higher when it has more number

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of insured customers with it and vice-versa. We also assume that the probability of joining is directly proportional to the number of customers.

The subsequent sections of this paper are arranged as follows - section 2 contains a brief review of the literature. Section 3 contains the description of the model. Sections 4 and 5 contain the derivation of steady state probabilities and performance measures considering the reneging rule as R_BOS and R_EOS respectively. Section 6 deals with the conclusions. Appendix A and Appendix B contains some derivations.

2. Literature Survey

One of the earliest works on balking was by Haight (1957). Barrer (1957) has carried out one of the early work on reneging where he considered deterministic reneging with single server Markovian arrival and service rates. Customers were selected randomly for service. In his subsequent work, Barrer (1957) also considered deterministic reneging (of both R_BOS and R_EOS type) in a multi-server scenario with FCFS discipline. The general method of solution was extended to two related queuing problems. Ancker and Gafarian (1963) have assumed the both reneging and balking in a M/M/1/N Markovian model. Rao (1965) has investigated the M/G/1 process subject to interruptions of service and where units balk and renege, and have obtained the steady state queue length probabilities and certain associated operational measures. Rao (1969) has considered the effect of post-ponable interruptions on a M/G/1 queuing process where customers balk with some probability and renege after having waited for a random length of time. Haghighi et al., (1986) have analyzed balking and reneging in a multi-server queuing model. Steady state distribution of the number of customers in the system was obtained. An expression for the average loss of customers during a fixed duration of time was also discussed.

Ke and Wang (1999) have done their work considering the M/M/R machine repair problem in which the failed machines balk with a constant probability $(1-b)$and reneging in which breakdown and repair time distributions of the servers follow the negative exponential distribution. They have also performed the sensitivity analysis part. Wang and Chang (2002) have derived the M/M/R queuing system with finite capacity considering balking, reneging and server breakdowns where the arriving customers balk with a probability $(1-b_n)$ and renege according to a negative exponential distribution. Sensitivity analysis of that model was also derived by them and they determined the optimal number of servers at minimum cost. Choudhury and Medhi (2011) have also analyzed a multiserver Markovian queuing system under the assumption that customers may balk as well as renege. They assumed that the reneging rate of the customers standing at a distance from the service facility is higher than those placed near the service facility i.e. reneging rate is a function of system state. State independent balking is considered with the assumption that each customer arriving at the system has a probability 'p' of balking from a non-empty queue. Explicit closed form expressions were presented. A numerical example with design aspects was also discussed to demonstrate results derived. Kuila (2013) has discussed about a steady state solution of the ordered queuing problem with balking and reneging. She has taken the waiting line is of chi-square queue with Poisson balking probability which depend not only on the number of customers in the system, but also the rate of services in the system. A very recent work that have done by Wang and Zhang (2018) where they have investigated an M/M/c queuing model impatience customers and dynamic service ability. Agarwal and Singh (2018) have illustrated the development of a queuing model consist of reneging and jockeying. In their work, the complex system includes three service terminals these all three were connected in parallel in tri-cum bi-serial way with a common service terminal.

Jain et al. (2014) developed and possibly introduced the concept of reverse balking in a single server Markovian queuing system having finite capacity. The concept of reverse balking evolves from its application in investment business. In such business, more number of customers associated with a firm becomes the attracting factor for investing customers and conversely. The steady-state solution of the model was obtained and different measures of effectiveness were derived. Sensitivity analysis of the model was also performed. Kumar et al. (2015) have assumed M/M/1/N feedback queuing system with reverse balking. Feedback customer in queuing literature refers to a customer who is unsatisfied with incomplete, partial or unsatisfactory service. The steady state solution and performance measures of the model have been computed. Som and Kumar (2017) have also considered a single server, finite capacity queuing system with customer retention and balking in which the inter-arrival and service times follow negative-exponential distribution. The reneging times were assumed to be exponentially distributed. An arriving customer may not join the queue if there was at least one customer in the system, i.e. the customer may balk. The steady state solution of the model has been obtained. Some performance measures have been computed. The sensitivity analysis of the model has been carried out. The effect of probability of retention on the average system size has been studied. The numerical results show that the average system size increases proportionately and steadily as the probability of retention increases. Some particular cases of the model have been derived and discussed.

3. Assumptions Of The Model

The model we deal with is based on the following assumptions:

a) Arrivals are described by Poisson probability distribution and the inter-arrival times are exponentially distributed with parameter λ. Arriving customers form a single queue.

b) There are 'k' servers and the service times are exponentially distributed with parameter μ. All Servers are homogenous and function independent of each other.

c) The capacity of the system is infinite.

d) The queue discipline is 'First-Come, First-served'.

e) Customers joining the system are assumed to be of Markovian reneging type. We shall assume that on joining the system, the customer is aware of its state in the system. Consequently, the reneging rate is modeled as a function of the customer's state in the system. In particular, a customer who is at state n will be assumed to have random patience time following exp (ν_n). Here, $\nu_n$is the reneging rate when state of the system is n. We considered ν_n under R-BOS as

   a. $\nu_n = \begin{cases} 0; n = 0,1,2, \dots \dots, k \\ \nu^{n-k}; n = k+1, k+2 \dots \dots \end{cases}$

   b. Here we also assume that $\nu > 1$. Our aim behind this formulation is to ensure that higher the current state of a customer, higher is the reneging rate. Hence we considered the reneging is position dependent.

f) We assume that balking is state dependent. It will be assumed that if the customer on arrival observes the system to be in state 'n', the probability that the customer will balk is p^{n-k} , where $n = k+1, k+2$ ...... i.e. the balking probability of the customer decreases as the state of the system goes up.

4. The Steady State Probabilites

In this section, the steady state probabilities are derived by using the Markov process method. Let $p_n$denotes the probability that there are 'n' customers in the system. The steady state equations are

$$\lambda p_0 = \mu p_1 \tag{1}$$

$$\lambda p_{n-1} + (n+1)\mu p_{n+1} = (\lambda + n\mu)p_n \quad ; \quad n = 1,2,\dots, k-1 \tag{2}$$

$$\lambda p_{k-1} + (k\mu + \nu)p_{k+1} = [k\mu + \lambda(1-p)]p_k \quad ; \quad n = k \tag{3}$$

$$\lambda(1-p^{n-k})p_{n-1} + \left[k\mu + \frac{\nu(\nu^{n-k+1}-1)}{(\nu-1)}\right]p_{n+1} = \left[\lambda(1-p^{n-k+1}) + \left(k\mu + \frac{\nu(\nu^{n-k}-1)}{(\nu-1)}\right)\right]p_n$$

$$n = k+1,\ k+2,\ \dots\dots,\infty \tag{4}$$

Solving recursively, we get

$$p_n = \left[\frac{\lambda^n}{n!\mu^n}\right]p_0 \quad ; n = 0,1,2,\dots, k \tag{5}$$

$$p_n = \left[\frac{\lambda^n \prod_{r=k+1}^{n}(1-p^{r-k})}{k!\mu^k \prod_{r=k+1}^{n}\left(k\mu + \frac{\nu(\nu^{r-k}-1)}{(\nu-1)}\right)}\right]p_0 \quad ; n = k+1,\ k+2,\ \dots\dots,\infty \tag{6}$$

Where p_0 is obtained from the normalizing condition $\sum_{n=0}^{\infty} p_n = 1$ and is given as

$$p_0 = \left[\frac{1}{1 + \sum_{n=1}^{k}\left[\frac{\lambda^n}{n!\mu^n}\right] + \sum_{n=k+1}^{\infty}\left[\frac{\lambda^n \prod_{r=k+1}^{n}(1-p^{r-k})}{k!\mu^k \prod_{r=k+1}^{n}\left(k\mu + \frac{\nu(\nu^{r-k}-1)}{(\nu-1)}\right)}\right]}\right]$$

Under R_EOS where customers may renege from queue as well as while being served. Let q_n denote the probability that there are *n* customers in the system. Applying Markov theory, we obtain the following set of steady state equations.

$$\lambda q_0 = (\mu + \nu) q_1 \tag{7}$$

$$\lambda q_{n-1} + (n+1)(\mu+\nu) q_{n+1} = \{\lambda + n(\mu+\nu)\} q_n \quad ; \qquad n = 1,2,\ldots, k-1 \tag{8}$$

$$\lambda q_{k-1} + \{k(\mu+\nu) + \nu\} q_{k+1} = [k(\mu+\nu) + \lambda(1-p)] q_k \tag{9}$$

$$\lambda(1-p^{n-k}) q_{n-1} + \left[k(\mu+\nu) + \frac{\nu(\nu^{n-k+1}-1)}{(\nu-1)}\right] q_{n+1}$$

$$=\left[\lambda(1-p^{n-k+1}) + \left(k(\mu+\nu) + \frac{\nu(\nu^{n-k}{}_{-1})}{(\nu-1)}\right)\right] q_n$$

$$n = k+1,\ k+2,\ \ldots\ldots,\infty \tag{10}$$

Solving recursively under R_EOS, we get

$$q_n = \left[\frac{\lambda^n}{n!(\mu+\nu)^n}\right] p_0 \;\; ; n = 0,1,2,\ldots, k \tag{11}$$

$$q_n = \left[\frac{\lambda^n \prod_{r=k+1}^{n}(1-p^{r-k})}{k!(\mu+\nu)^k \prod_{r=k+1}^{n}\left(k(\mu+\nu) + \frac{\nu(\nu^{r-k}{}_{-1})}{(\nu-1)}\right)}\right] q_0 \;\; ; n = k+1,\ k+2,\ \ldots\ldots,\infty \tag{12}$$

Where p_0 is obtained from the normalizing condition $\sum_{n=0}^{\infty} q_n = 1$ and is given as

$$q_0 = \left[\frac{1}{1 + \sum_{n=1}^{k}\left[\frac{\lambda^n}{n!(\mu+\nu)^n}\right] + \sum_{n=k+1}^{\infty}\left[\frac{\lambda^n \prod_{r=k+1}^{n}(1-p^{r-k})}{k!(\mu+\nu)^k \prod_{r=k+1}^{n}\left(k(\mu+\nu) + \frac{\nu(\nu^{r-k}{}_{-1})}{(\nu-1)}\right)}\right]}\right].$$

5. Performance Measures

The aim of all investigations in queueing theory is to get the main performance measures of the system which are the probabilistic properties of the following random variables: number of customers in the system, number of waiting customers, utilization of the server/s, response time of a customer, waiting time of a customer, idle time of the server, busy time of a server. Of course, the answers heavily depends on the assumptions concerning the distribution of inter arrival times, service times, number of servers, capacity and service discipline. It is quite rare, except for elementary or Markovian systems, that the distributions can be computed. Usually their mean or transforms can be calculated.

The average number of customers in the system is an important measure of system performance, which is denoted by L_s. Let *P(s)* be the p.g.f of the steady state probability. Then we note that

$$L_{s(R\_BOS)} = P'(1) = \frac{d}{ds} P(s) \,|_{s=1}$$

By solving this, we get (See the Appendix A and Appendix B for the derivations)

$$P'(s) = \frac{1}{\mu}\left[\lambda P(s) - \lambda p^{1-k} \sum_{n=k}^{\infty} p_n (ps)^n + \mu \sum_{n=k}^{\infty} n p_n s^{n-1} - \frac{1}{s(\nu-1)\nu^{k-1}} \sum_{n=k+1}^{\infty} p_n (\nu s)^n + \frac{\nu}{s(\nu-1)} \sum_{n=k+1}^{\infty} p_n s^n \right.$$

$$\left. - \frac{k\mu}{s} \sum_{n=k}^{\infty} p_n s^n \right]$$

Similarly under R_EOS, we get

$$Q^{/}(s) \ = \frac{1}{\mu+\nu}\left[\lambda Q(s)-\lambda p^{1-k}\sum_{n=k}^{\infty}q_n(ps)^n+(\ \mu+\nu)\sum_{n=k}^{\infty}nq_ns^{n-1}-\frac{1}{s(\nu-1)\nu^{k-1}}\sum_{n=k+1}^{\infty}q_n(\nu s)^n\right.$$
$$\left.+\frac{\nu}{s(\nu-1)}\sum_{n=k+1}^{\infty}q_ns^n-\frac{k(\mu+\nu)}{s}\sum_{n=k}^{\infty}q_ns^n\right]$$

Putting s=1 in the above equations, we get the average number of customers in the system i.e.

$$L_{s(R\_BOS)} \ =\frac{1}{\mu}\left[\lambda-\sum_{n=k}^{\infty}p_n(\lambda p^{n-k+1}-n\mu+k\mu)-\frac{1}{(\nu-1)}\sum_{n=k+1}^{\infty}p_n(\nu^{n-k+1}-\nu)\right]$$

$L_{s(R\_EOS)}$

$$=\frac{1}{\mu+\nu}\left[\lambda-\sum_{n=k}^{\infty}q_n\big(\lambda p^{n-k+1}-n(\mu+\nu)+k(\mu+\nu)\big)-\frac{1}{(\nu-1)}\sum_{n=k+1}^{\infty}q_n(\nu^{n-k+1}-\nu)\right]$$

Mean queue size can now be obtained and given by (under R_BOS)

$$L_{q(R\_BOS)}=\sum_{n=k+1}^{\infty}(n-k)p_n$$
$$=L_{s(R\_BOS)}-\sum_{n=1}^{k}np_n-k\left(\sum_{n=k}^{\infty}np_n-p_k\right)$$
$$=\frac{1}{\mu}\left[\lambda-\sum_{n=k}^{\infty}p_n(\lambda p^{n-k+1}-n\mu+k\mu)-\frac{1}{(\nu-1)}\sum_{n=k+1}^{\infty}p_n(\nu^{n-k+1}-\nu)-\mu\sum_{n=1}^{k}np_n-k\mu\left(\sum_{n=k}^{\infty}np_n-p_k\right)\right]$$

Similarly, Under R_EOS, we get

$$L_{q(R\_EOS)}=\sum_{n=k+1}^{\infty}(n-k)q_n$$
$$=L_{s(R\_EOS)}-\sum_{n=1}^{k}nq_n-k\left(\sum_{n=k}^{\infty}nq_n-q_k\right)$$
$$=\frac{1}{\mu+\nu}\left[\begin{array}{c}\lambda-\sum_{n=k}^{\infty}q_n\begin{pmatrix}\lambda p^{n-k+1}-n(\mu+\nu)\\+k(\mu+\nu)\end{pmatrix}-\frac{1}{(\nu-1)}\sum_{n=k+1}^{\infty}q_n(\nu^{n-k+1}-\nu)\\-(\mu+\nu)\sum_{n=1}^{k}nq_n-k\mu\left(\sum_{n=k}^{\infty}nq_n-q_k\right)\end{array}\right]$$

Using Little's formula, we can calculate the average waiting time in the system and average waiting time in queue from the above-mentioned mean lengths as follows:

Average waiting time in the system (W_s) is obtained as (under R_BOS)

$$W_{s(R\_BOS)}=\frac{L_{s(R\_BOS)}}{\lambda}$$
$$=\frac{1}{\lambda\mu}\left[\lambda-\sum_{n=k}^{\infty}p_n(\lambda p^{n-k+1}-n\mu+k\mu)-\frac{1}{(\nu-1)}\sum_{n=k+1}^{\infty}p_n(\nu^{n-k+1}-\nu)\right]$$

Under R_EOS,

$$W_{s(R\_EOS)} = \frac{L_{s(R\_EOS)}}{\lambda}$$

$$= \frac{1}{\lambda(\mu+\nu)}\left[\lambda - \sum_{n=k}^{\infty} q_n \begin{pmatrix} \lambda p^{n-k+1} - n(\mu+\nu) \\ +k(\mu+\nu) \end{pmatrix} - \frac{1}{(\nu-1)} \sum_{n=k+1}^{\infty} q_n (\nu^{n-k+1} - \nu)\right]$$

and average waiting time in queue (*W*) obtained as (Under R_BOS)

$$W_{q(R\_BOS)} = W_{s(R\_BOS)} - \frac{1}{\mu}$$

$$= \frac{1}{\lambda\mu}\left[\sum_{n=k}^{\infty} p_n(n\mu - \lambda p^{n-k+1} - k\mu) - \frac{1}{(\nu-1)} \sum_{n=k+1}^{\infty} p_n(\nu^{n-k+1} - \nu)\right]$$

Similarly, under R_EOS,

$$W_{q(R\_EOS)} = W_{s(R\_EOS)} - \frac{1}{\mu+\nu}$$

$$= \frac{1}{\lambda(\mu+\nu)}\left[\sum_{n=k}^{\infty} q_n(n(\mu+\nu) - \lambda p^{n-k+1} - k(\mu+\nu)) - \frac{1}{(\nu-1)} \sum_{n=k+1}^{\infty} q_n(\nu^{n-k+1} - \nu)\right]$$

Arrival rate of a customer into the system is λ. However, because of balking, all the customers who arrive do not join the system. Thus the effective arrival rate into the system is different from the overall arrival rate which is given by (R_BOS)

$$\lambda^e_{(R\_BOS)} = \lambda \sum_{n=0}^{k-1} p_n + \lambda \sum_{n=k}^{\infty} p_n(1 - p^{n-k+1})$$

Similarly, under R_EOS,

$$\lambda^e_{(R\_EOS)} = \lambda \sum_{n=0}^{k-1} q_n + \lambda \sum_{n=k}^{\infty} q_n(1 - p^{n-k+1})$$

We have assumed here that each customer has a random patience time following exp (ν_n) where $\nu_n$has been defined in Section 3. Reneging rate is a function of system state. Clearly then, the reneging rate of the system would depend on the state of the system as well as the reneging rule. The average reneging rates (*Avgrr*) is given by (R_BOS)

$$Avgrr_{R\_BOS} = \sum_{n=k+1}^{\infty} \frac{\nu(\nu^{n-k} - 1)}{(\nu-1)} p_n$$

Similarly, under R_EOS,

$$Avgrr_{R\_EOS} = \sum_{n=k+1}^{\infty} \frac{\nu(\nu^{n-k} - 1)}{(\nu-1)} q_n$$

In a real life situation, customers who balk or renege represent the business lost. Generally, customers are lost to the system in two ways, due to balking and due to reneging. Management would like to know the proportion of total customers lost in order to have an idea of total business lost.

Hence, the mean rate at which customers are lost under R_BOS is

$$\lambda - \lambda^e_{(R\_BOS)} + Avgrr_{R\_BOS} == \lambda - \mu \sum_{n=1}^{k} np_n - k\mu\left(\sum_{n=k}^{\infty} np_n - p_k\right)$$

Similarly, the mean rate at which customers are lost under R_EOS is

$$\lambda - \lambda^{e}_{(R_{EOS})} + Avgrr_{R_{EOS}} == \lambda - (\mu + \nu) \sum_{n=1}^{k} nq_n - k(\mu + \nu) \left(\sum_{n=k}^{\infty} nq_n - q_k \right)$$

These rates helps in determining the proportion of customers lost which is of interest to the system manager as also an important measure of business lost. This proportion is

$\{\lambda - \lambda^{e}_{(R\_BOS)} + Avgrr_{R\_BOS}\}/\lambda$ and $\{\lambda - \lambda^{e}_{(R\_EOS)} + Avgrr_{R\_EOS}\}/\lambda$

under R_BOS and R_EOS respectively. The proportion of customer completing receipt of service can now be easily determined from the above proportion.

6. Numerical Example

To illustrate the use of our results, we apply them to a queuing problem. We quote below an example from (Allen , 2005).

"KAMAKAZY AIRLINES is planning a new telephone reservation center. Each agent will have a reservation terminal and can serve a typical caller in 5 minutes, the service time being exponentially distributed. Calls arrive randomly and the system has a large message buffering system to hold calls that arrive when no agent is free. An average of 36 calls per hour is expected during the peak period of the day. The design criteria for the new facility is The probability a caller will find all agents busy must not exceed 0.1 (10%). How many agents (and terminals) should be provided?" (Table 1 should be here)

Table 1. Calculation of Difference Performance Measures

Performance measure	**Number of agents**		
	K=7	K=8	K=9
Probability that an arriving customer receive service immediately (=$\sum_{n=k+1}^{\infty} \boldsymbol{p_n}$)	0.15732	0.05229	0.0156
Probability that an arriving customer get all busy server (=$\mathbf{1} - \sum_{n=k+1}^{\infty} \boldsymbol{p_n}$)	0.84278	0.94771	0.9844
$\sum_{n=0}^{\infty} \boldsymbol{p_n}$	1	1	1
$\boldsymbol{p_0}$	0.050009	0.049853	0.049805
Proportion of customers lost due to reneging and balking	0.031352	0.011307	0.003656
Arrival rate of customers reaching service station ($= \boldsymbol{\lambda^s}$)	35.84149	35.94512	35.98274
Average number of customers in the system (= L)	3.06065	3.02748	3.00750

In order to meet the design requirement that the probability a caller will find all agents busy must not exceed 0.1 (10%), we observe from Table 1 that the number of agents should be 8 or more. Keeping cost consideration in view, fewer numbers of agents is better. We can thus settle for 8 agents. The proportion of customer lost is 1.1% where in the proportion of customer lost in (Choudhury and Medhi, 2011B) was 2.2%.The reason is that in (Choudhury and Medhi, 2011B), the balking probability was assumed to be constant whereas in our analysis, the probability of balking goes down as the system size increases. The probability of losing a customer is very small (1.1%) because the average length of the system is 3.02 in case the number of servers is 8 (i.e. k=8). Further the probability that an arriving customer receives service is 0.95. Thus 95% of customer do not balk or renege. Only 5% customers are lost (reneged or balked) and leave the system without service.

7. Conclusions

Analysis of a multi-server Markovian queuing system with state dependent balking and state dependent reneging has been presented. Closed form expressions of a number of performance measures have also been presented. To the best of our knowledge, this has not been attempted in literature. The assumptions of state dependent balking and state dependent reneging are the focus of this paper. Since balking and reneging is a commonly observed phenomenon, it is our belief that results presented in this work will be used by practitioners of queuing theory. One can obtain results of the traditional M/M/k models by substituting $\nu = 0$ and $p = 0$ in our results. An example from management prospective has been included to demonstrate the usefulness of our results.

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Appendix A

Derivation of $P^{/}(1)$ under R_BOS

Let *P(s)*denote the probability generating function, defined by

$$P^{/}(s) = \sum_{n=0}^{\infty} p_n s^n$$

From equation (2) we have

$\lambda p_{n-1} + (n+1)\mu p_{n+1} = (\lambda + n\mu)p_n$; $n = 1,2,\ldots, k\text{-}1$

Multiplying both sides of the equation by s^n and summing over 'n'

$$\lambda s\left[\sum_{n=1}^{k-1} p_{n-1}s^{n-1}\right] - \lambda\left[\sum_{n=1}^{k-1} p_n s^n\right] = \mu\sum_{n=1}^{k-1} np_n s^n - \frac{1}{s}\mu\sum_{n=1}^{k-1}(n+1)p_{n+1}s^{n+1} \quad \text{(a)}$$

From equation (3) we have

$$\lambda p_{k-1} + (k\mu + \nu)p_{k+1} = [n\mu + \lambda(1-p)]p_k$$

Multiplying both sides of the equation by s^k

$$\lambda s p_{k-1}s^{k-1} - \lambda(1-p)p_k s^k = k\mu p_k s^k - \frac{1}{s}(k\mu + \nu)p_{k+1}s^{k+1} \quad \text{(b)}$$

From equation (4) we have

$$\lambda(1-p^{n-k})p_{n-1} + \left[k\mu + \frac{\nu(\nu^{n-k+1}-1)}{(\nu-1)}\right]p_{n+1} = \left[\lambda(1-p^{n-k+1}) + \left(k\mu + \frac{\nu(\nu^{n-k}-1)}{(\nu-1)}\right)\right]p_n$$

$n = k+1,\ k+2,\ \ldots\ldots,\infty$

Multiplying both sides of the equation by s^n and summing over 'n'

$$\lambda s\left[\sum_{n=k+1}^{\infty}(1-p^{n-k})p_{n-1}s^{n-1}\right] - \lambda\left[\sum_{n=k+1}^{\infty}(1-p^{n-k+1})p_n s^n\right]$$
$$= \sum_{n=k+1}^{\infty}\left[k\mu + \frac{\nu(\nu^{n-k}-1)}{(\nu-1)}\right]p_n s^n - \frac{1}{s}\sum_{n=k+1}^{\infty}\left[k\mu + \frac{\nu(\nu^{n-k+1}-1)}{(\nu-1)}\right]p_{n+1}s^{n+1} \quad \ldots \text{(c)}$$

Now adding *(a)*, *(b)* and *(c)*

$$\Rightarrow \lambda s\left[\sum_{n=1}^{k-1} p_{n-1}s^{n-1} + p_{k-1}s^{k-1} + \sum_{n=k+1}^{\infty}(1-p^{n-k})p_{n-1}s^{n-1}\right] - \lambda\left[\sum_{n=1}^{k-1} p_n s^n + (1-p)p_k s^k + \sum_{n=k+1}^{\infty}(1-p^{n-k+1})p_n s^n\right] = \mu\sum_{n=1}^{k-1} np_n s^n + k\mu p_k s^k + \sum_{n=k+1}^{\infty}\left[k\mu + \frac{\nu(\nu^{n-k+1}-1)}{(\nu-1)}\right]p_n s^n - \frac{1}{s}\left[\mu\sum_{n=1}^{k-1}(n+1)p_{n+1}s^{n+1} + (k\mu+\nu)p_{k+1}{}^{k+1} + \sum_{n=k+1}^{\infty}\left[k\mu + \frac{\nu(\nu^{n-k}-1)}{(\nu-1)}\right]p_{n+1}s^{n+1}\right]$$

$$\Rightarrow \lambda s\left[\sum_{n=0}^{k-1} p_n s^n + \sum_{n=k+1}^{\infty} p_{n-1}s^{n-1} - p^{1-k}\sum_{n=k+1}^{\infty} p_{n-1}(ps)^{n-1}\right] - \lambda\left[\sum_{n=1}^{k-1} p_n s^n\right] - \lambda\left[\sum_{n=k}^{\infty} p_n s^n\right] + \lambda p^{1-k}\left[\sum_{n=k+1}^{\infty} p_n(ps)^n\right] = \mu s\left[P^{/}(s) - \sum_{n=k}^{\infty} np_n s^{n-1}\right] + k\mu\left[\mathrm{P}(s) - \sum_{n=0}^{k-1} p_n s^n\right] + \frac{\nu^{1-k}}{\nu-1}\left[\sum_{n=k+1}^{\infty} p_n(\nu s)^n\right] - \frac{\nu}{\nu-1}\left[\mathrm{P}(s) - \sum_{n=0}^{k} p_n s^n\right] - \frac{1}{s}\left[\mu s\left[P^{/}(s) - p_1 - \sum_{n=k+1}^{\infty} np_n s^{n-1}\right] + k\mu\left[\mathrm{P}(s) - \sum_{n=0}^{k} p_n s^n\right] + \nu p_{k+1}s^{k+1} + \sum_{n=k+1}^{\infty}\left(\frac{(\nu^{n-k+2}-\nu)}{(\nu-1)}\right)p_{n+1}s^{n+1}\right]$$

$$\Rightarrow \lambda s\{P(s)\} - \lambda s p^{1-k}\left[P(ps) - \sum_{n=0}^{k-1} p_n(ps)^n\right] - \lambda[P(s) - p_0] + \lambda p^{1-k}\left[P(ps) - \sum_{n=0}^{k-1} p_n(ps)^n\right] = \mu s\left[P^{/}(s) - \sum_{n=k}^{\infty} np_n s^{n-1}\right] + k\mu\left[\mathrm{P}(s) - \sum_{n=0}^{k-1} p_n s^n\right] + \frac{\nu^{1-k}}{\nu-1}\left[P(\nu s) - \sum_{n=0}^{k} p_n(\nu s)^n\right] - \frac{\nu}{\nu-1}\left[\mathrm{P}(s) - \sum_{n=0}^{k} p_n s^n\right] - \mu\left[P^{/}(s) - p_1 - \sum_{n=k+1}^{\infty} np_n s^{n-1}\right] - \frac{k\mu}{s}\left[\mathrm{P}(s) - \sum_{n=0}^{k} p_n s^n\right] - \frac{\nu}{s}p_{k+1}s^{k+1} - \frac{1}{s(\nu-1)\nu^{k-1}}\sum_{n=k+1}^{\infty} p_{n+1}(\nu s)^{n+1} + \frac{\nu}{s(\nu-1)}\sum_{n=k+1}^{\infty} p_{n+1}s^{n+1}$$

⇨ $\lambda s\{P(s)\} - \lambda s p^{1-k} P(ps) + \lambda s p^{1-k} \sum_{n=0}^{k-1} p_n (ps)^n - \lambda P(s) + \lambda p_0 + \lambda p^{1-k} P(ps) - \lambda p^{1-k} \sum_{n=0}^{k-1} p_n (ps)^n = \mu s P^{/}(s) - \mu s \sum_{n=k}^{\infty} n p_n s^{n-1} + k\mu P(s) - k\mu \sum_{n=0}^{k-1} p_n s^n + \frac{1}{(\nu-1)\nu^{k-1}} P(\nu s) - \frac{1}{(\nu-1)\nu^{k-1}} \sum_{n=0}^{k} p_n (\nu s)^n - \frac{\nu}{\nu-1} P(s) + \frac{\nu}{\nu-1} \sum_{n=0}^{k} p_n s^n - \mu P^{/}(s) + \lambda p_0 + \mu \sum_{n=k}^{\infty} n p_n s^{n-1} - k\mu p_k s^{k-1} - \frac{k\mu}{s} P(s) + \frac{k\mu}{s} \sum_{n=0}^{k-1} p_n s^n + \frac{k\mu}{s} p_k s^k - \frac{\nu}{s} p_{k+1} s^{k+1} - \frac{1}{s(\nu-1)\nu^{k-1}} \left[P(\nu s) - \sum_{n=0}^{k} p_n (\nu s)^n - p_{k+1} (\nu s)^{k+1}\right] + \frac{\nu}{s(\nu-1)} \left[P(s) - \sum_{n=0}^{k} p_n (s)^n - p_{k+1} (s)^{k+1}\right]$

⇨ $\lambda P(s)(s-1) - \lambda p^{1-k} P(ps)(s-1) + \lambda p^{1-k} (s-1) \sum_{n=0}^{k-1} p_n (ps)^n = \mu P^{/}(s)(s-1) - \mu \sum_{n=k}^{\infty} n p_n s^{n-1} (s-1) + k\mu P(s)(1-\frac{1}{s}) - k\mu(1-\frac{1}{s}) \sum_{n=0}^{k-1} p_n s^n + \frac{P(\nu s)}{(\nu-1)\nu^{k-1}} (1-\frac{1}{s}) - \frac{1}{(\nu-1)\nu^{k-1}} (1-\frac{1}{s}) \sum_{n=0}^{k} p_n (\nu s)^n - \frac{\nu}{\nu-1} P(s)(1-\frac{1}{s}) + \frac{\nu}{(\nu-1)} \left(1-\frac{1}{s}\right) \sum_{n=0}^{k} p_n (s)^n - \frac{p_{k+1} s^{k+1}}{s} \left[\nu + \frac{\nu}{(\nu-1)} - \frac{\nu^{1+k}}{(\nu-1)\nu^{k-1}}\right]$

⇨ $\lambda P(s)(s-1) - \lambda p^{1-k} P(ps)(s-1) + \lambda p^{1-k} (s-1)[P(ps) - \sum_{n=k}^{\infty} p_n (ps)^n] + \mu \sum_{n=k}^{\infty} n p_n s^{n-1} (s-1) - \frac{k\mu}{s} P(s)(s-1) - \frac{k\mu}{s} (s-1) \sum_{n=0}^{k-1} p_n s^n - \frac{P(\nu s)}{s(\nu-1)\nu^{k-1}} (s-1) + \frac{1}{s(\nu-1)\nu^{k-1}} (s-1) \sum_{n=0}^{k} p_n (\nu s)^n + \frac{\nu}{s(\nu-1)} P(s)(s-1) - \frac{\nu}{s(\nu-1)} (s-1) \sum_{n=0}^{k} p_n s^n = \mu P^{/}(s)(s-1)$

⇨ $\mu P^{/}(s) = \lambda P(s) - \lambda p^{1-k} \sum_{n=k}^{\infty} p_n (ps)^n + \mu \sum_{n=k}^{\infty} n p_n s^{n-1} - \frac{k\mu}{s} [P(s) - \sum_{n=0}^{k-1} p_n s^n] - \frac{1}{s(\nu-1)\nu^{k-1}} [P(\nu s) - \sum_{n=0}^{k} p_n (\nu s)^n] + \frac{\nu}{s(\nu-1)} [P(s) - \sum_{n=0}^{k} p_n s^n]$

⇨ $\mu P^{-}(s) = \lambda P(s) - \lambda p^{1-k} \sum_{n=k}^{\infty} p_n (ps)^n + \mu \sum_{n=k}^{\infty} n p_n s^{n-1} - \frac{k\mu}{s} \sum_{n=k}^{\infty} p_n s^n - \frac{1}{s(\nu-1)\nu^{k-1}} \sum_{n=k+1}^{\infty} p_n (\nu s)^n + \frac{\nu}{s(\nu-1)} \sum_{n=k+1}^{\infty} p_n s^n$

Now putting s=1, we get the average number of customers in the system i.e. L_s

⇨ $$L_s = P^{/}(1)$$

$$= \frac{1}{\mu}\left[\lambda - \lambda \sum_{n=k}^{\infty} p_n p^{n-k+1} + \mu \sum_{n=k}^{\infty} n p_n - k\mu \sum_{n=k}^{\infty} p_n - \frac{1}{(\nu-1)\nu^{k-1}} \sum_{n=k+1}^{\infty} p_n \nu^n + \frac{\nu}{(\nu-1)} \sum_{n=k+1}^{\infty} p_n\right]$$

$$= \frac{1}{\mu}\left[\lambda - \sum_{n=k}^{\infty} p_n (\lambda p^{n-k+1} - n\mu + k\mu) - \frac{1}{(\nu-1)} \sum_{n=k+1}^{\infty} p_n (\nu^{n-k+1} - \nu)\right]$$

Appendix B

Derivation of $Q^{/}(1)$ under R_EOS

Let *Q(s)* denote the probability generating function, defined by

$$Q^{/}(s) = \sum_{n=0}^{\infty} q_n s^n$$

From equation (8) we have

$\lambda q_{n-1} + (n+1)(\mu+\nu) q_{n+1} = \{\lambda + n(\mu+\nu)\} q_n$; *n = 1,2,…., k-1*

Multiplying both sides of the equation by s^n and summing over '*n*'

$$\lambda s \left[\sum_{n=1}^{k-1} q_{n-1} s^{n-1}\right] - \lambda \left[\sum_{n=1}^{k-1} q_n s^n\right]$$

$$= (\mu+\nu)\sum_{n=1}^{k-1} nq_n s^n - \frac{1}{s}(\mu+\nu)\sum_{n=1}^{k-1}(n+1)q_{n+1}s^{n+1} \quad \text{(d)}$$

From equation (9) we have

$$\lambda q_{k-1} + (k(\mu+\nu)+\nu)q_{k+1} = \{n(\mu+\nu)+\lambda(1-p)\}q_k$$

Multiplying both sides of the equation by s^k

$$\lambda s q_{k-1}s^{k-1} - \lambda(1-p)q_k s^k = k(\mu+\nu)q_k s^k - \frac{1}{s}(k(\mu+\nu)+\nu)q_{k+1}s^{k+1} \quad \text{(e)}$$

From equation (10) we have

$$\lambda(1-p^{n-k})q_{n-1} + \left[k(\mu+\nu) + \frac{\nu(\nu^{n-k+1}-1)}{(\nu-1)}\right]q_{n+1}$$

$$=\left[\lambda(1-p^{n-k+1}) + \left(k(\mu+\nu)+\frac{\nu(\nu^{n-k}-1)}{(\nu-1)}\right)\right]q_n$$

n = k+1, k+2, ……,∞

Multiplying both sides of the equation by s^n and summing over 'n'

$$\Rightarrow \lambda s\left[\sum_{n=k+1}^{\infty}(1-p^{n-k})q_{n-1}s^{n-1}\right] - \lambda\left[\sum_{n=k+1}^{\infty}(1-p^{n-k+1})q_n s^n\right] = \sum_{n=k+1}^{\infty}\left[k(\mu+\nu)+\frac{\nu(\nu^{n-k}-1)}{(\nu-1)}\right]q_n s^n - \frac{1}{s}\sum_{n=k+1}^{\infty}\left[k(\mu+\nu)+\frac{\nu(\nu^{n-k+1}-1)}{(\nu-1)}\right]q_{n+1}s^{n+1} \quad \text{(f)}$$

Now adding (d), *(e)* and *(f)*

$$\Rightarrow \lambda s\left[\sum_{n=1}^{k-1} q_{n-1}s^{n-1} + q_{k-1}s^{k-1} + \sum_{n=k+1}^{\infty}(1-p^{n-k})q_{n-1}s^{n-1}\right] - \lambda\left[\sum_{n=1}^{k-1} q_n s^n + (1-p)q_k s^k + \sum_{n=k+1}^{\infty}(1-p^{n-k+1})q_n s^n\right] = (\mu+\nu)\sum_{n=1}^{k-1} nq_n s^n + k(\mu+\nu)q_k s^k + \sum_{n=k+1}^{\infty}\left[k(\mu+\nu)+\frac{\nu(\nu^{n-k+1}-1)}{(\nu-1)}\right]q_n s^n - \frac{1}{s}\left[(\mu+\nu)\sum_{n=1}^{k-1}(n+1)p_{n+1}s^{n+1} + (k(\mu+\nu)+\nu)q_{k+1}{}^{k+1} + \sum_{n=k+1}^{\infty}\left[k(\mu+\nu)+\frac{\nu(\nu^{n-k}-1)}{(\nu-1)}\right]q_{n+1}s^{n+1}\right]$$

$$\Rightarrow \lambda s\left[\sum_{n=0}^{k-1} q_n s^n + \sum_{n=k+1}^{\infty} q_{n-1}s^{n-1} - p^{1-k}\sum_{n=k+1}^{\infty} q_{n-1}(ps)^{n-1}\right] - \lambda\left[\sum_{n=1}^{k-1} q_n s^n\right] - \lambda\left[\sum_{n=k}^{\infty} q_n s^n\right] + \lambda p^{1-k}\left[\sum_{n=k+1}^{\infty} q_n(ps)^n\right] = (\mu+\nu)s\left[Q'(s) - \sum_{n=k}^{\infty} nq_n s^{n-1}\right] + k(\mu+\nu)\left[Q(s) - \sum_{n=0}^{k-1} q_n s^n\right] + \frac{\nu^{1-k}}{\nu-1}\left[\sum_{n=k+1}^{\infty} q_n(\nu s)^n\right] - \frac{\nu}{\nu-1}\left[Q(s) - \sum_{n=0}^{k} q_n s^n\right] - \frac{1}{s}\left[(\mu+\nu)s\left[Q'(s) - q_1 - \sum_{n=k+1}^{\infty} nq_n s^{n-1}\right] + k(\mu+\nu)\left[Q(s) - \sum_{n=0}^{k} q_n s^n\right] + \nu q_{k+1}s^{k+1} + \sum_{n=k+1}^{\infty}\left(\frac{(\nu^{n-k+2}-\nu)}{(\nu-1)}\right)q_{n+1}s^{n+1}\right]$$

$$\Rightarrow \lambda s\{Q(s)\} - \lambda s p^{1-k}\left[Q(ps) - \sum_{n=0}^{k-1} q_n(ps)^n\right] - \lambda[Q(s) - q_0] + \lambda p^{1-k}\left[Q(ps) - \sum_{n=0}^{k-1} q_n(ps)^n\right] = (\mu+\nu)s\left[Q'(s) - \sum_{n=k}^{\infty} nq_n s^{n-1}\right] + k(\mu+\nu)\left[Q(s) - \sum_{n=0}^{k-1} q_n s^n\right] + \frac{\nu^{1-k}}{\nu-1}\left[Q(\nu s) - \sum_{n=0}^{k} q_n(\nu s)^n\right] - \frac{\nu}{\nu-1}\left[Q(s) - \sum_{n=0}^{k} q_n s^n\right] - (\mu+\nu)\left[Q'(s) - q_1 - \sum_{n=k+1}^{\infty} nq_n s^{n-1}\right] - \frac{k(\mu+\nu)}{s}\left[Q(s) - \sum_{n=0}^{k} q_n s^n\right] - \frac{\nu}{s}q_{k+1}s^{k+1} - \frac{1}{s(\nu-1)\nu^{k-1}}\sum_{n=k+1}^{\infty} q_{n+1}(\nu s)^{n+1} + \frac{\nu}{s(\nu-1)}\sum_{n=k+1}^{\infty} q_{n+1}s^{n+1}$$

$$\Rightarrow \lambda s\{Q(s)\} - \lambda s p^{1-k}Q(ps) + \lambda s p^{1-k}\sum_{n=0}^{k-1} q_n(ps)^n - \lambda Q(s) + \lambda q_0 + \lambda p^{1-k}Q(ps) - \lambda p^{1-k}\sum_{n=0}^{k-1} q_n(ps)^n = (\mu+\nu)sQ'(s) - (\mu+\nu)s\sum_{n=k}^{\infty} nq_n s^{n-1} + k(\mu+\nu)Q(s) - k(\mu+\nu)\sum_{n=0}^{k-1} q_n s^n + \frac{1}{(\nu-1)\nu^{k-1}}Q(\nu s) - \frac{1}{(\nu-1)\nu^{k-1}}\sum_{n=0}^{k} q_n(\nu s)^n - \frac{\nu}{\nu-1}Q(s) + \frac{\nu}{\nu-1}\sum_{n=0}^{k} q_n s^n - (\mu+\nu)Q'(s) +$$

$\lambda q_0 + (\mu + \nu) \sum_{n=k}^{\infty} n q_n s^{n-1} - k(\mu + \nu) q_k s^{k-1} - \frac{k(\mu+\nu)}{s} Q(s) + \frac{k(\mu+\nu)}{s} \sum_{n=0}^{k-1} q_n s^n + \frac{k(\mu+\nu)}{s} q_k s^k - \frac{\nu}{s} q_{k+1} s^{k+1} - \frac{1}{s(\nu-1)\nu^{k-1}} \left[Q(\nu s) - \sum_{n=0}^{k} q_n (\nu s)^n - q_{k+1} (\nu s)^{k+1}\right] + \frac{\nu}{s(\nu-1)} \left[Q(s) - \sum_{n=0}^{k} q_n s^n - q_{k+1} s^{k+1}\right]$

⇨ $\lambda Q(s)(s-1) - \lambda q^{1-k} Q(ps)(s-1) + \lambda p^{1-k}(s-1) \sum_{n=0}^{k-1} q_n (ps)^n = (\mu + \nu) Q'(s)(s-1) - (\mu + \nu) \sum_{n=k}^{\infty} n q_n s^{n-1}(s-1) + k(\mu + \nu) Q(s) \left(1 - \frac{1}{s}\right) - k(\mu + \nu)(1 - \frac{1}{s}) \sum_{n=0}^{k-1} q_n s^n + \frac{Q(\nu s)}{(\nu-1)\nu^{k-1}} (1 - \frac{1}{s}) - \frac{1}{(\nu-1)\nu^{k-1}} (1 - \frac{1}{s}) \sum_{n=0}^{k} q_n (\nu s)^n - \frac{\nu}{\nu-1} Q(s)(1 - \frac{1}{s}) + \frac{\nu}{(\nu-1)} \left(1 - \frac{1}{s}\right) \sum_{n=0}^{k} q_n s^n - \frac{q_{k+1} s^{k+1}}{s} \left[\nu + \frac{\nu}{(\nu-1)} - \frac{\nu^{1+k}}{(\nu-1)\nu^{k-1}}\right]$

⇨ $\lambda Q(s)(s-1) - \lambda p^{1-k} Q(ps)(s-1) + \lambda p^{1-k}(s-1)[Q(ps) - \sum_{n=k}^{\infty} q_n (ps)^n] + (\mu + \nu) \sum_{n=k}^{\infty} n q_n s^{n-1}(s-1) - \frac{k(\mu+\nu)}{s} Q(s)(s-1) - \frac{k(\mu+\nu)}{s} (s-1) \sum_{n=0}^{k-1} q_n s^n - \frac{Q(\nu s)}{s(\nu-1)\nu^{k-1}} (s-1) + \frac{1}{s(\nu-1)\nu^{k-1}} (s-1) \sum_{n=0}^{k} q_n (\nu s)^n + \frac{\nu}{s(\nu-1)} Q(s)(s-1) - \frac{\nu}{s(\nu-1)} (s-1) \sum_{n=0}^{k} q_n s^n = (\mu + \nu)(s-1) Q'(s)$

⇨ $(\mu + \nu) Q'(s) = \lambda Q(s) - \lambda p^{1-k} \sum_{n=k}^{\infty} q_n (ps)^n + (\mu + \nu) \sum_{n=k}^{\infty} n q_n s^{n-1} - \frac{k(\mu+\nu)}{s} [Q(s) - \sum_{n=0}^{k-1} q_n s^n] - \frac{1}{s(\nu-1)\nu^{k-1}} [Q(\nu s) - \sum_{n=0}^{k} q_n (\nu s)^n] + \frac{\nu}{s(\nu-1)} [Q(s) - \sum_{n=0}^{k} q_n s^n]$

⇨ $(\mu + \nu) Q'(s) = \lambda Q(s) - \lambda p^{1-k} \sum_{n=k}^{\infty} q_n (ps)^n + (\mu + \nu) \sum_{n=k}^{\infty} n q_n s^{n-1} - \frac{k(\mu+\nu)}{s} \sum_{n=k}^{\infty} q_n s^n - \frac{1}{s(\nu-1)\nu^{k-1}} \sum_{n=k+1}^{\infty} q_n (\nu s)^n + \frac{\nu}{s(\nu-1)} \sum_{n=k+1}^{\infty} q_n s^n$

Now putting s=1, we get the average number of customers in the system i.e. L_s

⇨ $L_s = Q'(1) = \frac{1}{(\mu+\nu)} \left[\lambda - \lambda \sum_{n=k}^{\infty} q_n p^{n-k+1} + (\mu + \nu) \sum_{n=k}^{\infty} n q_n - k(\mu + \nu) \sum_{n=k}^{\infty} q_n - \frac{1}{(\nu-1)\nu^{k-1}} \sum_{n=k+1}^{\infty} q_n \nu^n + \frac{\nu}{(\nu-1)} \sum_{n=k+1}^{\infty} q_n\right] = \frac{1}{(\mu+\nu)} \left[\lambda - \sum_{n=k}^{\infty} q_n \left(\lambda p^{n-k+1} - (n-k)(\mu + \nu)\right) - \frac{1}{(\nu-1)} \sum_{n=k+1}^{\infty} q_n (\nu^{n-k+1} - \nu)\right]$