# Stochastic Maximum Flow Network Interdiction with Endogenous Uncertainty

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**Abstract**

We described the two-stage maximum flow network interdiction problem under endogenous stochastic interdiction. Our model consists of two adversary agent playing a Stackelberg game. A smuggler who wishes to maximize the expected flow of some illicit commodities (same as drugs), can be transmitted between a source node and a sink node without being detected. On the other hand, an attacker tries to minimize the objective of the smugglers by installing some detectors or adding some security controls on critical arcs to increase the probability of detection. Most previous stochastic network interdiction problems in the literature deal with exogenous uncertainty, while we consider stochastic programs under endogenous uncertainty in which the interdictor's decisions can alter the probability measures. The problem can be formulated as a bi-level program, at the top level the attacker by a limited budget, choosing critical arcs to install detectors and enhance the interdiction probability of those arcs endogenously. The bottom level problem is a two-stage problem which is solved to find the maximum flow in the network by smugglers. In the first stage, he chooses some links to transmit the flow. In the second stage an indicator variable is used to show if he would be detected under each scenario. The bi-level decomposition algorithm has been applied to solve the problem by adding some Benders' cuts iteratively. We applied a successive method, to deal with non-linearity rise in the probability measure of each path. A case study of drug trafficking network is applied to recognize which countries have the most significant effect in interdicting the drug trafficking network. The police can concentrate on those areas to decline the amount of drug flow. Our results demonstrate that if the critical arcs are chosen wisely and the probability of drug seizers decreases slightly, a significant decrease in the expected total flow of drugs can be achieved.

**Keywords:** Network Interdiction; Maximum Flow; Endogenous Uncertainty; Benders' Decomposition; Stackelberg Game.

## 1. Introduction

In the defense context, *interdiction* refers to actions that serve to block or otherwise inhibit an adversary's operations and often regards attacks against supply chain operations or communications (Smith and Song 2019). A network interdiction problem usually is formulated as Stackelberg games (von Stackelberg 1952) involving two opponent players who compete in a min-max or max-min game. One player, defender, wishes to optimize its objective over the network, for example, maximum flow (Wollmer 1964),(Altner and Uhan 2009), the shortest path (Israeli and Wood 2002), or maximum coverage of the demand in a facility interdiction problem (Church, Scaparra, and Middleton 2004, Akbari-Jafarabadi et al. 2017). The opponent, the interdictor, uses a limited budget to alter the network by attacking some links or nodes to maximally impair the defender's objective.

In this study, we tried to investigate interdiction problems when the objective is maximum flow. In maximum flow network interdiction, the defender seeks to maximize the flow from a specified origin to a given destination on a directed and capacitated network, while the attacker's goal is to minimize his opponent's maximum flow by interdicting some arcs.

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In some situations, an unfavorable or illegal flow of material is transmitted via a network, and then an intervention plan is needed to interrupt the flow in the network. These problems are studied in the literature, for example, drug trafficking (Malaviya, Rainwater, and Sharkey 2012a), hospital infection (Assimakopoulos 1987), nuclear smuggling (Morton, Pan, and Saeger 2007), illegal immigrants(Zhang, Zhuang, and Behlendorf 2018) or flood control (Soleimani-Alyar, Ghaffari-Hadigheh).

Wollmer (1964) proposed the first interdiction problem and modeled a min-max flow in a network by removing a given number of arcs. The network interdiction problem in general case is NP-complete (Wood 1993) and a polynomial time algorithm for interdiction problem over planar graphs was presented in (Pan and Schild 2016). The network interdiction problem has received some attention in recent decades because of the wide range of applications varying across supply chains (Liberatore, Scaparra, and Daskin 2011), military planning (Scaparra and Church 2008), toll control (Borndörfer, Sagnol, and Schwartz 2016), nuclear smuggling (Michalopoulos, Barnes, and Morton 2014), drug enforcement (Malaviya, Rainwater, and Sharkey 2012a), water resource management (Qiao et al. 2007) and (Soleimani-Alyar, Ghaffari-Hadigheh, and Sadeghi 2016), and electrical grid analysis (Salmeron, Wood, and Baldick 2004).

Recently, some researchers extended the basic form of network interdiction problem and studied different forms of these problems including path-based interdiction (Granata, Steeger, and Rebennack 2013); interdiction with asymmetric information (Bayrak and Bailey 2008); the bi-objective and tri-objective interdiction (Rocco S. and Ramirez-Marquez 2010), (Chen, Guo, and Yu 2019) and (Ramirez-marquez, A, and S 2010); the multi-commodity networks (Lim and Smith 2007); the tri-level network interdiction with fortification (Sadeghi, Seifi, and Azizi 2017); dynamic network interdiction (Rad and Kakhki 2013,Jabbarzare, Zolfagharinia, and Najafi 2019); and stochastic network interdiction (Ramirez-Marquez and Rocco S. 2009),(Cormican, Morton, and Wood 1998) and (Morton, Pan, and Saeger 2007). Two comprehensive surveys about network interdiction models and algorithms are provided in (Smith and Song 2019 , Smith, Prince, and Geunes 2013).

From the certainty point of view, the network interdiction optimization problem is divided into two classes; including deterministic optimization and stochastic programming. Wood (1991) studied the deterministic network interdiction with binary interdiction. The stochastic variants of network interdiction problems have received more attention recently. The first stochastic network interdiction problem with uncertain and binary interdiction success was formulated by Cormican et al. (1996). the objective of the problem was to minimize the expected maximum flow (Cormican, Morton, and Wood 1998). The presence of uncertainty in both traveling cost and interdiction impacts is studied in Song, Yongjia (2016) where a solution method based on branch and cut algorithm is applied. A stochastic shortest path network interdiction with a fixed probability of detection is addressed in Zhang, Zhuang, and Behlendorf (2018). The stochastic form of network flow interdiction with Stochastic under heterogeneous risk preferences is studied by Lei, Shen, and Song (2018). The other uncertain element which is considered for network interdiction problems is the number of attacks (Liberatore, Scaparra, and Daskin 2011), origin-destination pair(Morton, Pan, and Saeger 2007)and the mass and type of shipping material (Nehme 2009).

In some real applications, uncertainty is endogenous, meaning that the decisions can influence and change the probability measures related to different scenarios. Most stochastic interdiction models assume exogenous uncertainty in which the probability of events is fixed and given. Our model is concerned with the maximum flow interdiction problem under endogenous uncertainty with decision dependent probabilities. We studied a bi-level programming problem. In the first level, the interdictor chooses some critical links to enhance interdiction ability by installing some detection sensors. The choice of links will change the associated probability of detection scenarios. In the second level, the defender will decide how to flow an illicit material (for example drugs) in the network via a two-stage stochastic program. In a "here and now situation", he has to find a maximum flow in the network without any information on the occurring scenarios. Then, in the second stage, a control variable is used to indicate whether or not he is detected under various scenarios. We assumed a prior probability of detection associated with each link and this probability will change with the decision on installing the detection sensor. In the sequel, we arrived at a stochastic interdiction model of maximum flow network with endogenous probabilities. A Benders' decomposition approach is then used to solve the proposed bi-level programming model. A real-world case study of worldwide drug trafficking network is presented herein.

The rest of this paper is structured as follows. Section 2 introduces the notation and presents the basic formulation of a bi-level maximum flow network interdiction problem. Then, the proposed stochastic interdiction problem is described. The linearization method is applied in Section 3. The solution method is applied in Section 4. Computational results on a real-world case study are presented in Section 5. Finally, we draw some concluding remarks in Section 6.

## 2. Problem description and formulations

In this section, we describe a bi-level programming formulation of maximum flow network interdiction problem, then the developed stochastic version of the problem with endogenous uncertainty will be introduced.

### 2.1. Bi-level Maximum Flow Network Interdiction

The maximum flow network interdiction problem is defined in a directed graph $G(N, A)$, with nodes set $N$ and arcs set $A$. The interdiction problem is a leader-follower, Stackelberg game. The problem is between two parties known as defender and attacker (or interdictor). The defender (follower) seeks to maximize the flow from origin node $O$ to destination node $D$. The interdictor (leader) goal is to discover some critical arcs and attack the network due to limited budget $B$ to minimize the defender's maximum flow. Let $u_k$ denote the capacity of arc $k \in A$ and dummy arc $(D, O)$ with capacity $u_{DO} = \infty$. The bi-level maximum flow network interdiction is formulated as follows:

$$\min_{x \in X} \max_{y} \quad y_{DO} \qquad\qquad \text{DMFI}^1 \text{ Model}$$

$$s.t. \quad \sum_{k \in FS(i)} y_k - \sum_{k \in RS(i)} y_k = o \quad \forall i \in N, (i, j) = k$$

$$y_k \leq u_k(1 - x_k) \quad \forall k \backslash \{FS(O), RS(D)\}$$

Variable set $x \in \{0,1\}^{|A|}$ represents the attack vector and $y \geq 0$ is the flow vector. The feasible set $X$ is defined as $X = \{x \in \{0,1\}^{|A|} \mid c^T x \leq B\}$ where $c_k$ is the cost of interdicting, arc k $\in A$. $FS(i) = \{j \in N \mid (i, j) \in A\}$ is the forward star (set of arcs going out of node $i$) and $RS(i) = \{j \in N \mid (j, i) \in A\}$ is the reverse star (set of arcs going into node $i$) of each node $i \in N$.

### 2.2. Uncertain maximum flow network interdiction

It is clear that interdiction problems are further complicated in the presence of uncertainty. However, in some real applications, two players of the game (defender and attacker) are not fully cognizant of their opponent and should decide under uncertainty. In this study, we consider the situation that the probability of a successful attack is decision dependent and changes endogenously in the model.

For the uncertain version of the interdiction problem, suppose that the interdiction action for any arc $k \in A$ may be successful with the given probability measure $p_k$ ($0 \leq p_k \leq 1$) initially and the interdictor can increase this probability measure to $q_k$ ($0 \leq q_k \leq 1$ and $q_k \geq p_k$) by enhancing the interdiction ability. This enhancement may be achieved by adding some sensors or security controls according to the interdictor's limited budget. Due to the limited interdiction budget $B$, the interdictor tries to find the best possible arcs of the network to enhance the interdiction ability, in order to maximize the defender's traversing cost.

The problem can be formulated as a bi-level program which involves two nested optimization models in which the upper agent optimizes its own objective function subject to the reaction of the agent in a subsequent level. At the top level, the attacker tries to identify critical arcs and enhance the interdiction probability of those arcs from $p_k$ to $q_k$ for damaging the network and increasing the cost of traversing. The bottom level is a two-stage stochastic problem by itself. The first stage is solved to find the maximum flow in the damaged network and in the second stage and after scenario realization, the follower recognizes if the flow is detected by the leader or not.

**Problem notations:**

**Network components**

| | |
|---|---|
| $G(N, A)$ | Weighted and directed network with nodes set $N$ and arcs set $A$ |
| $FS(i)$ | Set of arcs going out of node $i$ |
| $RS(i)$ | Set of arcs going into node $i$ |

**Input parameter**

| | |
|---|---|
| $B$ | The total interdiction budget |
| $c_k$ | The cost of interdicting arc $k \in A$ |
| $C_{max}$ | The penalty which is imposed when no allowed flow exists |
| $p_k$ | The probability of successful detection on arc $k \in A$ with no interdiction action on that arc |
| $q_k$ | The probability of successful detection on arc $k \in A$ with interdiction action on that arc |

**Random elements**

| | |
|---|---|
| $\mathcal{S} \in \{0,1\}^{|A|}$ | Sample space |
| $s = (b_1, \dots, b_{|A|}) \in \mathcal{S}$ | A scenario where $b_k^s$ is a binary parameter that is 1 if the defender being interdicted while transmitting flow in the arc $k$ and 0 otherwise |
| $\mathcal{P}_x^s$ | The probability measure related to scenario $s \in \mathcal{S}$ depends on variable $x$ |
| $\pi_n^s$ | The probability measures induced by decision vector $x$ under scenario $s \in \mathcal{S}$. |

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[1] Deterministic Maximum Flow Interdiction

**variables**

| | |
|---|---|
| $x_k$ | Interdictor's decision variable. $x_k = 1$ if arc $k \in A$ is interdicted and $x_k = 0$ otherwise |
| $y_k$ | Evaders' decision variable. $y_k = 1$ if arc $k \in A$ is traversed and $y_k = 0$ otherwise |
| $y_{DO}$ | The total flow from origin node $O$ to destination node $D$ |
| $\gamma^s$ | The indicator variable which is 1 if the evader traverse arc $k \in A$ where $b_k^s = 1$ |

Now, we outline the stochastic model. For each arc, $k \in A$ defines a binary decision variable $x_k$ which is 1 if the defender enhances the interdiction ability and 0 otherwise. $y_k$ is defined as a bottom level variable which indicates the flow of arc $k \in A$. The total flow is $y_{DO}$ if an O-D flow exists due to scenario $s \in S$ where $S$ is the set of all possible scenarios. If an O-D flow is not possible, it means the flow is blocked by an interdicted arc, and the defender is penalized with a fixed number $C_{max}$. A binary indicator variable $\gamma^s$ is defined to indicate if there is a successful O-D flow or not. The $\gamma^s$ is 1 if the defender flows the material in an interdicted arc in scenario $s \in S$ and 0 otherwise.

The objective function of the problem is formulated as, $\sum_s \mathcal{P}^s \sum_{k \in A}[(1 - \gamma^s)y_{DO} - \gamma^s C_{max}]$, where $\mathcal{P}_x^s$ is the probability measure related to scenario $s \in S$ depending on decision $x$. The maximum flow network interdiction model with the above objective function and related constraints are presented as follows:

$$\min_x \; \sum_s \mathcal{P}_x^s \sum_{k \in A}[(1 - \gamma^s)y_{DO} - \gamma^s C_{max}] \tag{1}$$

$$s.t.$$

$$\sum_{k \in A} c_k x_k \leq B \tag{2}$$

$$x_k \in \{0,1\} \quad \forall k \in A$$

$$\max_{y,\gamma} \; \sum_s \mathcal{P}_x^s \sum_{k \in A}[(1 - \gamma^s)y_{DO} - \gamma^s C_{max}] \tag{3}$$

$$s.t. \tag{4}$$

$$\sum_{k \in FS(i)} y_k - \sum_{k \in RS(i)} y_k = o \quad \forall i \in V \backslash \{s, t\}$$

$$y_k \leq u_k \qquad \forall k \in A \tag{5}$$

$$\gamma^s \geq \frac{1}{n} \sum_{(k \in A):b_k^s = 1} y_k \qquad \forall s \in S \tag{6}$$

$$\gamma^s \leq \sum_{(k \in A):b_k^s = 1} y_k \qquad \forall s \in S$$

$$y_k \in \{0,1\} \qquad \forall k \in A \quad , \gamma^s \in \{0,1\} \qquad \forall s \in S \tag{7}$$

Constraint (2) shows budget limitation, constraint set (4) and (5) stand for flow conservation and capacity limitations. The constraint set (6) linearized the subsequent phrase:

if $\exists k \in A : (y_k = 1 \text{ and } b_k^s = 1)$ then $\gamma^s = 1$ and vice versa

The equation $\gamma^s \geq \frac{1}{n} \sum_{(k \in A):b_k^s = 1} y_k$, guarantees that if there is at least one arc $k$ in which $y_k = 1$ while $b_k = 1$ then $\gamma^s$ is forced to be 1, otherwise it takes the value zero, where $b_k^s$ is a binary parameter that is 1 if the defender interdicted the arc $k$ and 0 otherwise. The upper-level player prefers $\gamma^s = 1$, because in this situation the bottom level player is interdicted and should pay the penalty $C_{max}$ and the upper-level objective function is $\max_x \; \sum_s \mathcal{P}_x^s \gamma^s C_{max}$. On the other hand, $\gamma^s = 0$ is favorable for the defender (bottom level player) and he achieves the objective $\max_{y,\gamma} \sum_s \mathcal{P}^s \sum_{k \in A}(1 - \gamma^s)d_k y_k$ in this condition.

Here we explain the bi-level and two-stage programming definition to distinguish them. Bi-level programs are mathematical programs with embedded optimization problems in their constraints. The main problem is called the upper-level problem or the leader and the nested problem is called the lower-level problem or the follower. A famous example in economics is the Stackelberg game. Two-stage programming is a stochastic program. In the first stage or "here and know" stage, a decision maker should decide before the realization of the uncertain data. After fixing the optimal solution of the first stage and when uncertain parameters take their values, decision maker decides on recourse action in the second stage and the optimal solution is determined.
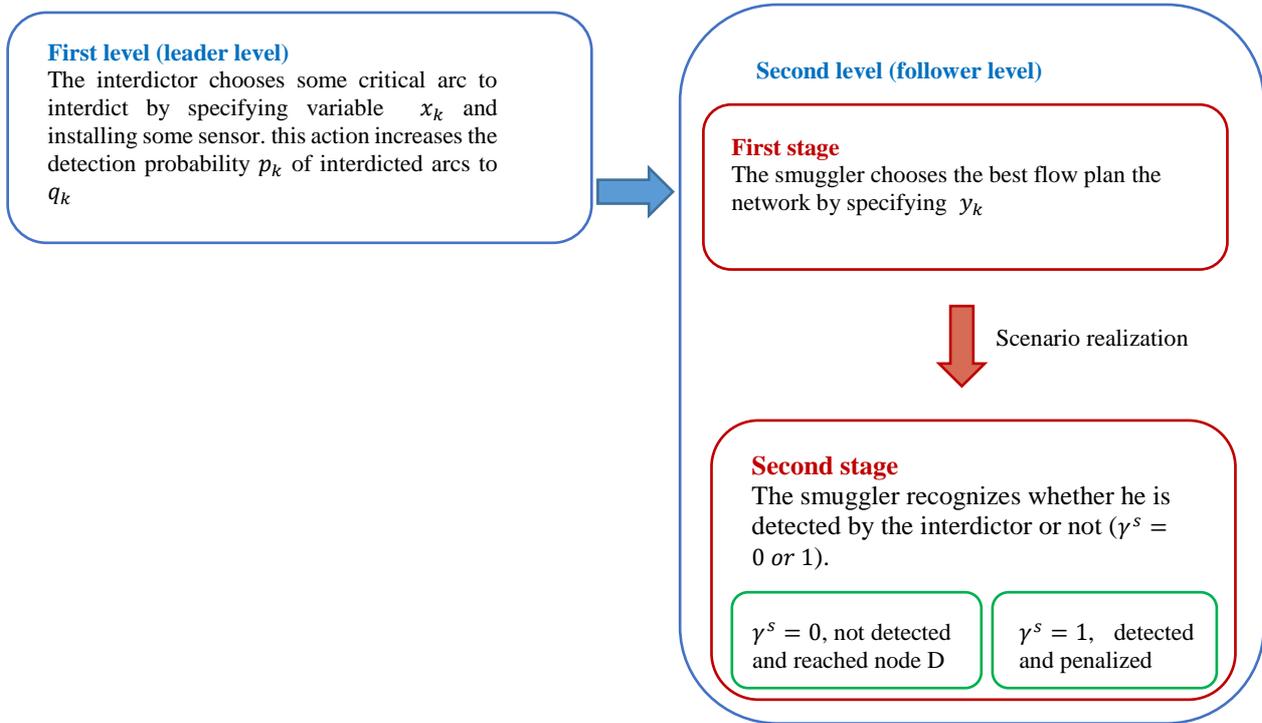
**Figure 1.** The decision process for model 1.

### 3) Dealing with Non-linear Probability Measures

As discussed in section 2, the uncertain interdiction problem is modeled as a bi-level and two-stage stochastic program. In the upper level, the attacker decides which arcs to use to enhance interdiction ability by setting $x_k = 1$. In the bottom level, first, the defender tries to find the maximum flow in the survived network and then an indicator variable reveals whether he flows the material to terminal node successfully or if he is detected.

In the objective function, the expected amount of flow should be computed over all scenarios. $\mathcal{P}_x^s$, the probability measure related to scenario $s \in \mathcal{S}$ can be expressed as a function of the upper-level decision vector $x_k$.

$$\mathcal{P}_x(\{b_k = \tilde{b}_k\}) = \begin{cases} (p_k(1 - x_k) + q_k x_k) & b_k^s = 1 \\ [((1 - p_k)(1 - x_k) + (1 - q_k)x_k)] & b_k^s = 0 \end{cases}$$

Then it can be represented as:

$$\mathcal{P}_x(\{b_k = \tilde{b}_k\}) = \prod_{k \in A} [b_k^s (p_k(1 - x_k) + q_k x_k)] + [(1 - b_k^s)((1 - p_k)(1 - x_k) + (1 - q_k)x_k)]$$

Since the interdictor decisions ($x_k = 1 \text{ or } 0$) affect the probability measures, the problem has endogenous uncertainty. As you see, the above phrase includes the product of decision variables, then using such products results in non-linearity. The most common approaches to deal with this nonlinearity are linearizations or convex approximations. Laumanns et al. (Laumanns, Prestwich, and Kawas 2014) developed a novel and useful method based on Bayse' rule to find the simple linear relationship between resulting probability measures, called "neighboring measures".

Let $|A| = n$. Suppose that all arcs are sorted by an ordinal relation, then the set of arcs can be shown by $A = \{1, 2, \dots, n\}$. Two decision vectors $x^0, x^1 \in \{0,1\}^{|E|}$ of the arcs set $A$, are just different in one component corresponding to arc $k$ therefore $x^1 - x^0 = \mathbb{e}_k$ where $\mathbb{e}_k$ is the $k - th$ unit vector. So that two probability measures $x^0$ and $x^1$ are interrelated as:

$$\mathcal{P}_{x^1}(\{b_k = \tilde{b}_k\}) = \begin{cases} (\mathcal{P}_{x^0}).\dfrac{q_k}{p_k} & for \ b_k^s = 1 \\ (\mathcal{P}_{x^0}).\dfrac{1 - q_k}{1 - p_k} & for \ b_k^s = 0 \end{cases}$$

$\mathcal{P}_{x^1}$ is the probability measure of scenario $s$ associated with decision vector $x^1$. The term, $x^1 - x^0 = \mathbb{e}_k$ means that $x_1^k$ =1 while $x_0^k = 0$ assumes that $b_k^s = 1$, the arc $k \in A$, is not interdicted in $x^0$, then the detection probability of this arc is $p_k$. on the other hand, this arc is interdicted in decision vector $x^1$ then the associate probability should change to $q_k$, multiplying $\mathcal{P}_{x^0}$ to $\frac{q_k}{p_k}$ substituting $q_k$ to $x_1^k$.

$$\mathcal{P}_{x^0} = (..., p_k, ...) \underset{\times \frac{q_k}{p_k}}{\Longrightarrow} (..., q_k, ...) = \mathcal{P}_{x^1}$$

The following example can clarify the method. Consider the simple complete graph.1 with 3 arcs ($|A| = 3$) and 3 nodes. The decision vector $x^0 = (0,0,0)$ means the interdictor does not have any enhancement plan for arcs and the decision vector $x^1 = (1,0,0)$ displays an enhancement plan established in the second arc by the interdictor.
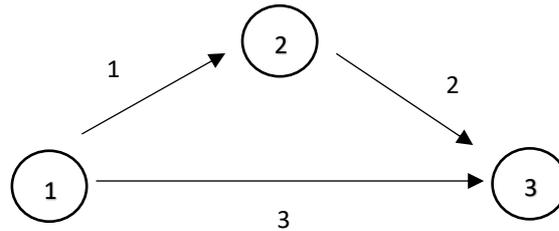


**Graph 1.** A simple graph with $|A| = |N| = 3$

Arc $k$ interdicts successfully with probability $p_k = 0.6$ if $x_k = 0$ (no enhancement plan in arc $k$) and with probability $q_k = 0.8$ if arc $k$ selects to enhance, so that $x_k = 1$.

Assuming $p_k = 0.6$ and $q_k = 0.8$, the probability measures of different scenarios under two decision vectors $x^0 = (0,0,0)$ and $x^1 = (1,0,0)$ are presented in Table.1.

**Table1.** Probability measures for neighboring decision vectors $x^0$ and $x^1$

| $b_k^s$ | | | $P_x\{\xi = \tilde{\xi}\}$ | | |
|---|---|---|---|---|---|
| 1 | 2 | 3 | $x^0 = (0,0,0)$ | | $x^1 = (0,0,1)$ |
| 0 | 0 | 0 | 0.064 | *0.2/0.4 | 0.032 |
| 0 | 0 | 1 | 0.096 | *0.8/0.6 | 0.128 |
| 0 | 1 | 0 | 0.096 | *0.2/0.4 | 0.048 |
| 0 | 1 | 1 | 0.144 | *0.8/0.6 | 0.192 |
| 1 | 0 | 0 | 0.096 | *0.2/0.4 | 0.048 |
| 1 | 0 | 1 | 0.144 | *0.8/0.6 | 0.192 |
| 1 | 1 | 0 | 0.144 | *0.2/0.4 | 0.072 |
| 1 | 1 | 1 | 0.216 | *0.8/0.6 | 0.288 |

In the last row of the table, the probability of $x^0 = (0,0,0)$, $\mathcal{P}_{x^0}$, is $p_k \times p_k \times p_k = 0.6 \times 0.6 \times 0.6 = 0.216$. On the other hand, $\mathcal{P}_{x^1} = p_k \times p_k \times p_k \cdot \left(\frac{q_k}{p_k}\right) = p_k \times p_k \times q_k = 0.6 \times 0.6 \times 0.8 = 0.288$.

The mentioned method can be applied to our optimization model. In other words, instead of decision vector $x$ which is determined in the model, defining successive vectors $\hat{x}^k$ for a fixed $x$ and all $k \in \{1,2,...,n\}$, such that:

$$\tilde{x}_i^k = \begin{cases} x_i & if\ i \leq k \\ 0 & else \end{cases} \quad k \in 1,...,n$$

Then, we have $\hat{x}^n = x$. Defining the probability measure induced by $\hat{x}^k$ as $\pi_k^s = \mathcal{P}_{\hat{x}^k}$ which is an auxiliary variable, for any $k \leq n$, two neighbored probability measures $\pi_k^s$ and $\pi_{k-1}^s$ must satisfy the following linear inequalities:

$$\begin{cases} \pi_k^s \leq \dfrac{q_k}{p_k} . \pi_{k-1}^s + 1 - x_k & \forall s : b_k^s = 1 \\ \pi_k^s \leq \dfrac{1-q_k}{1-p_k} . \pi_{k-1}^s + 1 - x_k & \forall s : b_k^s = 0 \end{cases}$$

$$\{\pi_k^s \leq \pi_{k-1}^s + x_k \quad \forall s\}$$

$$\sum_s \pi_k^s = 1$$

For the initial vector $\hat{x}^0 = (0, \dots, 0)$, the associated probability measure is a function of $p_k$ and is defined as $\pi_0^s = \prod_{k: b_k^s=0}(1 - p_k) . \prod_{k: b_k^s=1} p_k$ , besides the final vector $\pi_n^s$ , is the probability measure induced by decision vector $x$ under scenario $s \in \mathcal{S}$.

Now we replace the probability measure $\mathcal{P}_x^s$ by the variable $\pi_n^s$ and add the corresponding successive constraint to model (1)-(7) and construct model (8)-(14). Briefly, $\pi_n^s$ is the probability of an evader being detected under scenarios $\in \mathcal{S}$. It is a probability measure that is induced by the decision vector $x$, which shows whether or not a sensor is installed on a link.

$$\min_{x,\pi} \max_{y,\gamma} \sum_s \pi_n^s \sum_{k \in A} [(1 - \gamma^s)y_{DO} + \gamma^s C_{max}] \tag{8}$$

$$s.t. \tag{9}$$

$$\sum_{k \in A} c_k x_k \leq B$$

$$x_k \in \{0,1\} \quad \forall k \in A$$

$$\pi_k^s \leq \frac{q_k}{p_k} \times \pi_{k-1}^s + 1 - x_k \qquad \forall k \in A, \forall s : b_k^s = 1 \tag{10}$$

$$\pi_k^s \leq \frac{1-q_k}{1-p_k} \times \pi_{k-1}^s + 1 - x_k \qquad \forall k \in A, \forall s : b_k^s = 0$$

$$\pi_k^s \leq \pi_{k-1}^s + x_k \qquad \forall k \in A, \forall s \in S$$

$$\sum_s \pi_k^s = 1 \qquad \forall k \in A$$

$$\pi_k^s \geq 0 \qquad \forall k \in A, \forall s \in S$$

$$\sum_{k \in FS(i)} y_k - \sum_{k \in RS(i)} y_k = 0 \quad \forall i \in V \backslash \{s, t\} \tag{11}$$

$$y_k \leq u_k \qquad \forall k \in A \tag{12}$$

$$\gamma^s \geq \frac{1}{n} \sum_{(k \in A): b_k^s=1} y_k \qquad \forall s \in S \tag{13}$$

$$\gamma^s \leq \sum_{(k \in A): b_k^s=1} y_k \qquad \forall s \in S$$

$$y_k \in \{0,1\} \qquad \forall k \in A \tag{14}$$

$$\gamma^s \in \{0,1\} \qquad \forall s \in S$$

Model (8)-(14) is a max-min problem with objective function (8) and constraint set (9) to (14). (9) Shows the budget limitation; (10) presents the set of inequality to linearize the product caused by probability measures; (11) and (12) are the constraint of flow conservation and capacity limitation; (13) is constraint related to indicator variable; and lastly (14) display the binary form of variables. Although, the term $\gamma^s y_k$ causes nonlinearity in Model 2, we can simply linearize it as a product of two binary variables.

### 4) Solution method

Model (8)-(14) is a bi-level programming model and there are three main approaches to solve such models, including decomposition, duality, and reformulation. In this section, we show how the problem displayed in Model 2 can be solved using Benders' based bi-level decomposition.

The decomposition approach is established on solving a smaller nested bi-level programming problem sequentially. This method lets one exploit the interaction between the attacker and the defender in each of these sub-problems. A given

initial solution for the top-level problem is needed to start, then an equilibrium for the bottom level problems is found iteratively and a Benders' cut is generated by the obtained solution. The new cut should be added to the top-level problem in the next iteration and solves the upper level considering the set of all generated cuts. Model (15)-(16) is the reformulation of Model (8)-(14) to apply the decomposition method. (16) is the Benders' cut generated using the fixed value $\hat{\gamma}^s$ and $\hat{y}_k$.

$$\min_{\alpha,\pi,x} \alpha \tag{15}$$

$s.t.$

Constraints (9) and (10)

$$\alpha \geq \sum_{s} \pi_n^s \sum_{k \in A} [(1 - \hat{\gamma}^s)\hat{y}_{DO} + \hat{\gamma}^s C_{max}]$$

$$\alpha(\tilde{\pi}, \tilde{x}) = \max_{y,\gamma} \sum_{s} \tilde{\pi}_n^s \sum_{k \in A} [(1 - \gamma^s)y_{DO} + \gamma^s C_{max}] \tag{16}$$

$s.t.$

Constraints (11) to (14)

Let $S(\tilde{y}, \tilde{\gamma})$ denotes the set of pairs $(\tilde{y}, \tilde{\gamma})$ found as the optimal solutions of the lower level problem. Algorithm 1 to solve Model (15)-(16) is stated as:

Algorithm 1:
Initialization phase: Given an O-D network and a tolerable optimality gap $\mathcal{E}$ ; set $\tilde{w}^{(1)} \leftarrow \mathbf{0}$ , $S(\tilde{y}, \tilde{\gamma}) = \emptyset$.
Iterative phase:
For $i = 1, .., N$ ,
    Begin
      1.1) Solve (16) to find $\tilde{y}^i, \tilde{\gamma}^i$ and objective value $\tilde{Z}^i$. Set $S(\tilde{y}, \tilde{\gamma}) = S(\tilde{y}, \tilde{\gamma}) \cup \{\tilde{y}^i, \tilde{\gamma}^i\}$
      1.2) Solve (15) to find $\tilde{x}^i, \tilde{\pi}^i$ and $\tilde{\alpha}^i$.
      1.3) (Convergence Check) If $|\tilde{\alpha}^i - \tilde{Z}^i| < \varepsilon$ then $(x^*, \pi^*, y^*, \gamma^*) \leftarrow (\tilde{x}^i, \tilde{\pi}^i, \tilde{y}^i, \tilde{\gamma}^i)$ and break.
                        else go to 1.1.
    End.
Return $x^*, \pi^*, y^*, \gamma^*$.

## 5. Computational Results of the Case Study

To illustrate the importance and applicability of the proposed uncertain maximum flow network interdiction a real-world case study is presented in the following.

### 5.1. Case study: Drug Trafficking Network

In October 2009, UNODC[2] published a survey on the transnational threat of Afghan drug (opium) and revealed the dramatic negative effect of Afghan drugs not only in neighboring states but also all over the world. The report emphasizes the importance of seizing the drug flow between different countries. Figure 1 illustrated the global drug trafficking graph.

Now, to apply our model to cope with the drug trafficking problem, each country may consider a node of graph $G$ and a link between two countries if there is drug transportation.

To simplify, we just study some weighty and critical links and consider Afghanistan for the origin of graph $G$ as the main producer of opium in the world. Europe with the largest share of opiate market value is regarded as a single destination. A directed and weighted graph $G = (N, A)$ with 10 arcs $A$ and 7 nodes $N$ is presented in figure 2, which depicts the selected opium flow network.

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[2] United Nations Office of Drug and Crime

**Figure1.** Estimated opium flows from Asia to the world (in tons)*



**Figure 2.** Summarized graph derived from figure1

Table 2 clarifies the drug flows (tons) after 2002 for 7 selected countries with a drastic role in worldwide flow. The percentage of estimated flow intercepted in each country is also shown.

**Table 2.** Percentage of seizers and estimated flows of selected graph

| # | country | Percentage of seizers | Afghanistan | Pakistan | Central Asia | Iran | Russia | Turkey | Europe |
|---|---------|-----------------------|-------------|----------|--------------|------|--------|--------|--------|
| 1 | Afghanistan | 2% | | 150 | 95 | 105 | | | |
| 2 | Pakistan | 17% | | | | 35 | | | 5 |
| 3 | Central Asia | 5% | | | | | 77 | | |
| 4 | Iran | 20% | | | 2 | | | 95 | |
| 5 | Russia | 2% | | | | | | | 4 |
| 6 | Turkey | 9% | | | | | | | 85 |
| 7 | Europe | 9% | | | | | | | |

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* source: UNODC

### 5.2. Numerical results and discussion

We implement Algorithm 1 using a simplified instance derived from UNODC report 2009. The algorithm is implemented in GAMS software installed on a PC with an Intel Core i7m 4 GHz processor and 16 GB of RAM. The commercial solver CPLEX, version 11, was used to solve the MIP problems at each node of the enumeration tree.

We executed the model using Algorithm 1, for $p$ as a percentage of seizers (shown in table 3) or probability of successful detection before interdiction, and $q = 1$ (problem #1) $q = p + uniform(0, 1 - p)$ (problem #2), $q = 0.5$ (problem #3) and $q = p + 0.3$ (problem #4). $q$ is the probability of successful detection which is enhanced after interdiction. This enhancement may be achieved by adding some sensors or security enhancing.

Table 3 represents the results. $x$ is the master problem binary variable associated with interdicted arcs and $y$ is the binary variables demonstrating the chosen path traversed by the smugglers. The optimal value of the min-max problem is shown under column $Z$.

**Table 3.** Optimal results and computational time of different problem instances for Budget $B = 1$ to $4$

| Problem No. | $B$ | $x$ | $y$ | $z$ | TIME (s) |
|---|---|---|---|---|---|
| #1 | 1 | 8 | 1-2-5-6-9 | 3.60 | 28.38 |
| | 2 | 6,8 | 1-5 | 2.38 | 93.25 |
| | 3 | 1,2,3 | 1-2-3-4-5-6 | -6 | 91.01 |
| | 4 | 1,2,3 | 7-8-9-10 | -6 | 92.37 |
| | | | | | |
| #2 | 1 | 1 | 3-8-10 | 20.06 | 4.03 |
| | 2 | 1,8 | 2-6-9 | 2.47 | 24.05 |
| | 3 | 1,6,8 | 1-5 | 0.88 | 145.24 |
| | 4 | 1,2,5,8 | 3-6-7-9 | -0.58 | 397.15 |
| #3 | | | | | |
| | 1 | 8 | 3-8-10 | 8.32 | 14.17 |
| | 2 | 8,10 | 1-2-5-6-9 | 3.60 | 89.38 |
| | 3 | 6,8,10 | 1-5 | 2.31 | 392.48 |
| | 4 | 5,6,8,10 | 3-8-10 | 1.88 | 476.38 |
| #4 | | | | | |
| | 1 | 3 | 1-4-8-10 | 11.05 | 20.97 |
| | 2 | 3,8 | 1-4-8-10 | 0.25 | 37.43 |
| | 3 | 3,8,10 | 3-8-10 | 5.89 | 86.07 |
| | 4 | 3,6,8,10 | 3-8-10 | 5.89 | 86.07 |

To analyze the number of the table, consider the optimal value of problem instance #4 (numbers in the grey rectangle). If the network protector enhances the prior detection probability measure $p$ to $p + 0.3$ just for a single link ($B = 1$), the optimal value ( the expected flow that the smugglers may pass through the network) is 11.05, but if there is enough budget to amplify the detection probability of 2 arcs, the optimal value decreases 10.80 units and reaches to 0.25. In addition, our model finds critical links to invest. Here, link 3, 8 are recognized as critical links that need to enhance the detection probability. Link 8 which is associated with Iran-Turkey arc appeared frequently in different solutions and show the critical role of this connection in the flow control.

The optimal value of different problem instances #1 to #4 for associated budget $B = 1$ to $B = 4$ is illustrated in figure 3.

| | B=1 | B=2 | B=3 | B=4 |
|---|---|---|---|---|
| ▪ q=1 | 3/598 | 2/308 | -6 | -6 |
| ▪ q=p+U(0,1-p) | 20/058 | 2/472 | 0/881 | -0/578 |
| ▪ q=0.5 | 8/317 | 3/598 | 2/308 | 1/867 |
| ▪ q=p+0.3 | 11/046 | 9/248 | 5/897 | 5/897 |
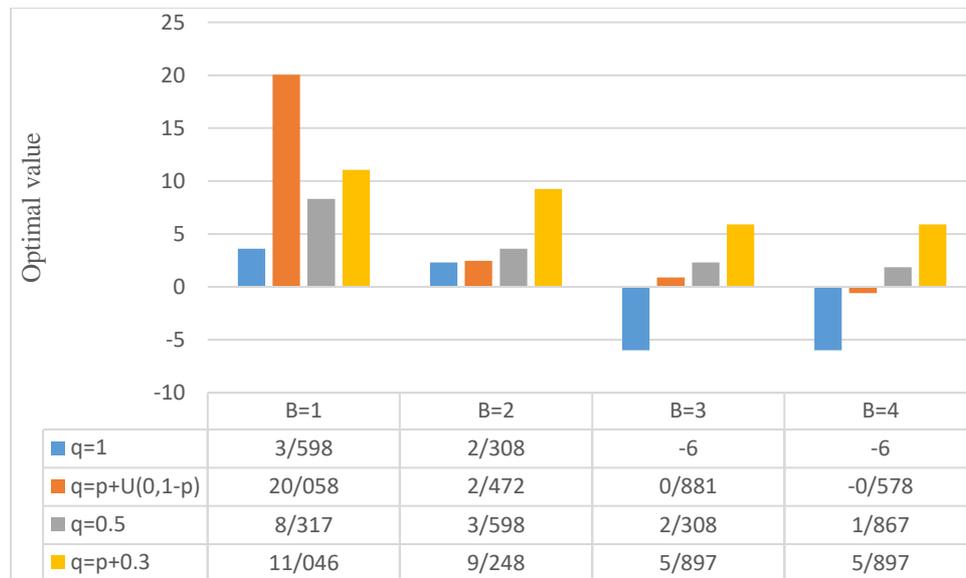
**Figure 3.** Optimal values of different problems instances for $B = 1 \; to \; 4$

As expected, the optimal value of the min-max interdiction problem decrease by having more budget $B$. This decline standstill for extra budget and stop decreasing after a threshold. Figure 2 shows this optimal value trend for problem instances #2 and #3.
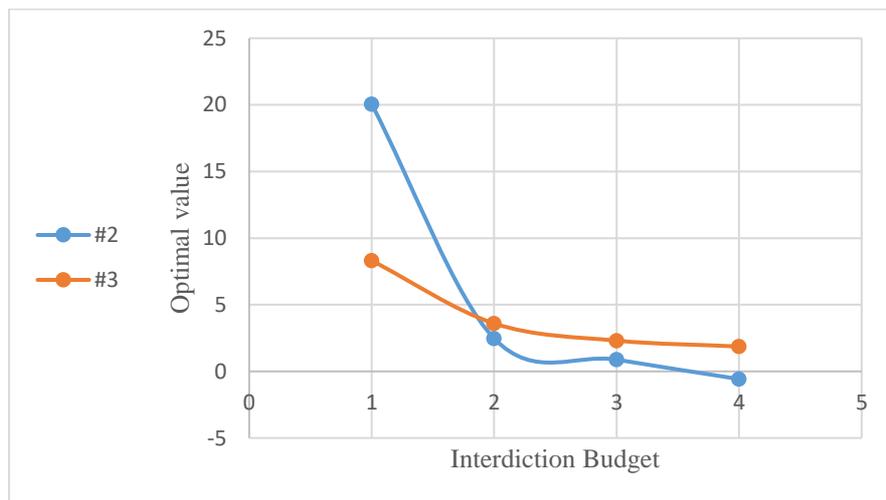


**Figure 4.** Optimal values of problem #2 for $B = 1 \; to \; 6$

## 6. Conclusions

We have studied two-stage maximum flow network interdiction problem with decision-dependent probabilities which change endogenously in the model. In this problem, endogenous uncertainty arises in this way. Considering that the interdiction action for any arc $k \in A$ may be successful with the given probability $p_k$ $(0 \leq p_k \leq 1)$ initially, the interdictor can increase this probability to $q_k$ $(0 \leq q_k \leq 1$ and $q_k \geq p_k)$ by enhancing the interdiction ability. This enhancement may be achieved by adding some sensors or security controls according to the interdictor's limited budget. The proposed model is a nested bi-level programming model in which the leader chooses some arcs to enhance the probability of detection and probability of each scenario as a consequent at the upper level. At the bottom level, the follower tries to choose the best possible flow plan having just the probability of each scenario. On the other hand, the problem is a two-stage stochastic problem. In the first stage, the follower selects a flow plan while he does not know which scenario may occur. In the second stage, a control value reveals if the follower is detected finally or there is a successful flow of commodity from origin to destination.

We proposed a new formulation and solution method for a bi-level maximum flow problem with endogenous uncertainty. The problem is formulated with a bi-level min-max structure, which could not be solved via standard optimization algorithms. In this paper, bi-level decomposition algorithm has been applied to solve the problem by adding some Benders' cuts iteratively. We applied a method, called distribution shaping, to deal with non-linearity arise in the probability measure of each flow plan. The model is implemented for a real case study of selected sub-graph derived from opium trafficking network. The network topology and data are extracted from the UNODC report (2009).

Numerical results show that a small increase in the probability of opium seizers leads to a significant change in the expected total cost of smugglers. In addition, our model determines the most important countries in the case study to watch for controlling the world flow of opium.

The two-stage problem consists of an exponential number of scenarios where solving the model in a larger size is difficult or even impossible. Using a scenario reduction method that is consistent with endogenous uncertainty may be interesting for future work.

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