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Hyperbolic Cosine – Exponentiated Exponential Lifetime Distribution and its Application in Reliability

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Abstract
Recently, Kharazmi and Saadatinik (2016) introduced a new family of lifetime distributions called hyperbolic cosine – F (HCF) distribution. In the present paper, it is focused on a special case of HCF family with exponentiated exponential distribution as a baseline distribution (HCEE). Various properties of the proposed distribution including explicit expressions for the moments, quantiles, mode, moment generating function, failure rate function, mean residual lifetime, order statistics and expression of the entropy are derived. Estimating parameters of HCEE distribution are obtained by eight estimation methods: maximum likelihood, Bayesian, maximum product of spacings, parametric bootstrap, non-parametric bootstrap, percentile, least-squares and weighted least-squares. A simulation study is conducted to examine the bias, mean square error of the maximum likelihood estimators. Finally, one real data set has been analyzed for illustrative purposes and it is observed that the proposed model fits better than Weibull, gamma and generalized exponential distributions.

Keywords:
Hyperbolic cosine function; Exponentiated exponential; Hazard function; Mean residual lifetime; Maximum likelihood estimates.

1. Introduction

In last three decades or so, an extensive research works have appeared in the literature on the theory of statistical distributions. Statistical models are commonly applied to describe real world phenomena. Classical distributions have been extensively used for modeling data in several areas such as engineering, actuarial, environmental and medical sciences, biological studies, demography, economics, finance and insurance. However, in many applied areas such as lifetime analysis, finance and insurance, there is a clear need for extended forms of these distributions. For that reason, several methods for generating new families of distributions have been studied. The well-known generators are the following:: Azzalini's skew family by Azzalini (1985), Marshal-Olkin generated family (MO-G) by Marshall and Olkin (1997), exponentiated family (EF) of distributions by Gupta et al. (1998), beta-G by Eugene et al. (2002)and Jones (2004), Kumaraswamy-G (Kw-G) by Cordeiro and de Castro (2011) McDonald-G (Mc-G) by Alexander et al. (2012), gamma-G (type 1) by Zografos and Balakrishnan (2009), gamma-G (type 2) by Ristić and Balakrishnan (2012), gamma-G (type 3) by Torabi and Hedesh (2012), log-gamma-G by Amini et al.(2014), logistic-G by Tahir et al. (2016), exponentiated generalized-G by Cordeiro et al. (2013), geometric exponential-Poisson family by Nadarajah et al. (2013a), truncated-exponential skew-symmetric family by Nadarajah et al. (2014), logistic-generated (Lo-G) family by Torabi and Montazari (2014), Transformed-Transformer (T-X) by Alzaatreh et al. (2013), exponentiated (T-X) by Alzaghal et al. (2013), Weibull-G by Bourguignon et al. (2014), Exponentiated half logistic generated family by Cordeiro et al. (2016), Kumaraswamy Odd log-logistic-G by Alizadeh et al. (2015b), Kumaraswamy Marshall-Olkin by

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Alizadeh et al. (2015c), Beta Marshall-Olkin by Alizadeh et al. (2015a), Type Half-Logistic family of distributions by Cordeiro et al. (2015) and Odd generalized exponential-G by Tahir et al. (2015). These families of distributions have received a great deal of attention in recent years.

Kharazmi and Saadatinik (2016) introduced a new family of distributions by using the hyperbolic cosine function. This new class of distributions is obtained by compounding a baseline F distribution with the hyperbolic cosine function. This technique resulted in adding an extra parameter to a family of distributions for more flexibility. The hyperbolic cosine function has similar name to the trigonometric functions, but it is defined in terms of the exponential function as follows:

$$cosh(x) = \frac{e^{x} + e^{-x}}{2}.$$

This function is even and has a Taylor series expression with only even exponents for x as

$$cosh(x) = \sum_{n=0}^{\infty} \frac{x^{2n}}{(2n)!}. \tag{1}$$

Throughout the present paper, we use series expression (1) to obtain statistical and reliability properties of the HSF family of distributions. According to Kharazmi and Saadatinik (2016) a random variable X has a hyperbolic cosine-F (HCF) distribution if its cumulative distribution function (CDF) is given by

$$G(x) = \frac{2\, e^{\mathrm{a}}}{\mathrm{e}^{2\mathrm{a}} - 1} \sinh(\mathrm{a}\, F(x)), \tag{2}$$

Where$x > 0$, $a > 0$. In relation (3) the baseline $F(x)$ can be the CDF of any random variable. Kharazmi and Saadatinik (2016) assumed $F(x) = 1 - e^{-\lambda x}$ (the exponential distribution function). This new model is called HCE by them. Several interesting properties of this distribution have been established by authors. Moreover, it is investigated that the HCE model has increasing, decreasing and upside-down bathtub shaped hazard rate functions.

It is well known that the exponentiated exponential (EE) Gupta and Kundu (1999) distribution is one of the most widely used lifetime distribution in reliability engineering and other disciplines. However, the EE distribution does not exhibit a bathtub or upside-down bathtub shaped hazard rate function and thus it cannot be used to model the complex lifetime of a system. Hence a number of extensions of the EE distribution are introduced to overcome this shortage.

The aim of the present paper is to introduce hyperbolic cosine-exponentiated exponential (HCEE) model and study its different properties. The rest of the paper is organized as follows. In Section 2, we introduce a new model so-called HCEE model and discuss some general properties of this distribution. In Section 3, we obtain some statistical and reliability functions of HCEE model. We discuss different estimation procedures of unknown parameters in Section 4. In Section 5, a Monte Carlo simulation study is conducted to examine the bias and mean square error of the maximum likelihood estimators for each parameter. The analysis of one real data set has been presented in Section 6. Finally in Section 7 we conclude the paper.

2. Hyperbolic cosine – exponentiated exponential (HCEE) distribution

Consider the generalized exponential (GE) (or exponentiated exponential (EE)) distribution of Gupta and Kundu (1999) with the CDF $F(x) = (1 - e^{-\lambda x})^{\beta}$.

Definition1: The random variable X is said to have the hyperbolic cosine exponentiated exponential (HCEE) diostribution, if the PDF of X is

$$g(x, a, \beta, \lambda) = \frac{2\mathrm{a}\, e^{\mathrm{a}}}{\mathrm{e}^{2\mathrm{a}} - 1} \beta\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\beta - 1} \cosh(\mathrm{a}\big(1 - e^{-\lambda x}\big)^{\beta}) \tag{3}$$

where$x > 0$, $a > 0$, $\beta > 0$, $\lambda > 0$. We will denote it as $HCEE(a, \beta, \lambda)$. Fig 1 shows the shapes of $HCEE(a, \beta, \lambda)$ for different values of parameters. Clearly, changing parameters of model cause different skewness in PDF. So this model can be applied in many applications.

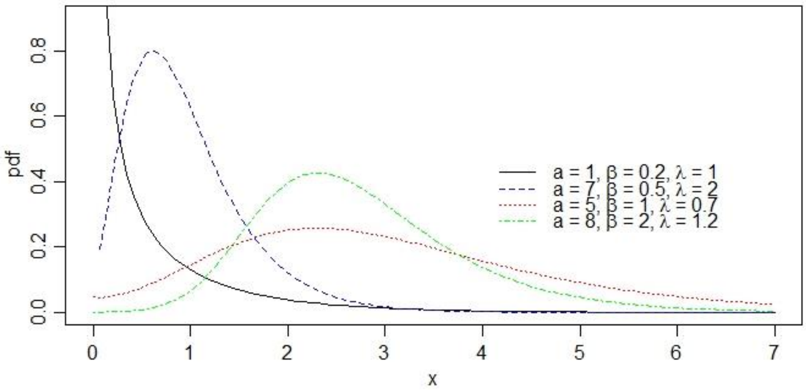

Figure 1. Plots of the $HCW(a, \beta, \lambda)$ for different values of parameters.

An interesting motivation for introducing this new model is an application in reliability. Suppose that the failure of a device occurs due to the presence of an unknown number, $2N+1$, of initial defects of some kind. Let $Y_1, \ldots, Y_{2N+1}$ denote the failure times of the initial defects. Let X denote the failure time of the device. Then $X = max(Y_1, \ldots, Y_{2N+1})$. Suppose N is a discrete random variable with a probability mass function:

$$P(N=n) = \begin{cases} \dfrac{2e^{a}}{e^{2a}-1}\dfrac{a^{2n+1}}{(2n+1)!} & n = 0,1,2,\ldots \\ 0 & o.w. \end{cases}$$

Where $0 < a < \infty$. Suppose also that $Y_1, \ldots, Y_{2N+1}$ is a random sample from the Weibull distribution with pdf $f(x) = \beta\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\beta-1}$ and cdf $F(x) = (1-e^{-\lambda x})^{\beta}$, then

$$f_{X|N=n}(x) = (2n+1)f(x)F^{2n}(x),$$

So the marginal probability density function of X is given by (3).
On the other hand, using the series expansion (1), the HCEE distribution can be stated as a mixtures of exponentiated$-F$ ($F^{\alpha}(x)$) distributions as follows:

$$g(x) = \frac{2\mathrm{a}\, e^{\mathrm{a}}}{\mathrm{e}^{2\mathrm{a}}-1} f(x)\cosh(\mathrm{a}\, F(x)) = \sum_{n=0}^{\infty} w(a,n) f_U(x),$$

Where $U \sim$ exponentiated $-F$ as

$$f_U(x) = (2n+1)f(x)F^{2n}(x)$$

And $w(a,n) = \frac{2ae^{a}}{e^{2a}-1}\frac{a^{2n}}{(2n+1)!}$.

3. Statistical and reliability properties

In this section, we study several statistical and reliability properties of the HCEE distribution, such as the survival function (SF), conditional survival function (CSF), failure rate (or hazard) function (FR), moment generating function (MGF), mean residual life (MRL) time and jth moment.

3.1. Survival, quantile, conditional reliability and failure rate function

The CDF of HCEE using (3) can be written as

$$G(x, a, \beta, \lambda) = \frac{2\, e^{\mathrm{a}}}{\mathrm{e}^{2\mathrm{a}}-1}\sinh(\mathrm{a}(1-e^{-\lambda x})^{\beta}). \tag{4}$$

Also, survival and quantile functions are simply given by

$$\bar{G}(x,a,\beta,\lambda) = 1 - G(x,a,\beta,\lambda) = 1 - \frac{2\,e^{a}}{e^{2a}-1}\sinh(a(1-e^{-\lambda x})^{\beta})\,, \quad (5)$$

$$x_p = -\frac{1}{\lambda}\left(\log\left(1-\left(\frac{\operatorname{arcsinh}\left(\frac{e^{2a}-1}{2e^{a}}p\right)}{a}\right)^{\frac{1}{\beta}}\right)\right) =$$

$$-\frac{1}{\lambda}\left(\log\left(1-\left(\frac{\log\left(\frac{e^{2a}-1}{2e^{a}}p+\sqrt{\left(\frac{e^{2a}-1}{2e^{a}}p\right)^{2}+1}\right)}{a}\right)^{\frac{1}{\beta}}\right)\right), \quad 0 \le p \le 1$$

the last equation comes from this fact that $\operatorname{arcsinh}(x) = \log(x+\sqrt{x^2+1})$. Also conditional reliability function is given by

$$\bar{G}(x,a,\beta,\lambda|t) = \frac{\bar{G}(x+t,a,\beta,\lambda)}{\bar{G}(t,a,\beta,\lambda)} = \frac{1-\frac{2\,e^{a}}{e^{2a}-1}\sinh(a(1-e^{-\lambda(x+t)})^{\beta})}{1-\frac{2\,e^{a}}{e^{2a}-1}\sinh(a(1-e^{-\lambda(x+t)})^{\beta})}\;; x>0\,, t>0.$$

From (3) and (4) it is easy to verify that the failure rate function is given by

$$h(x,a,\beta,\lambda) = \frac{\frac{2a\,e^{a}}{e^{2a}-1}\beta\lambda e^{-\lambda x}(1-e^{-\lambda x})^{\beta-1}\;\cosh(a(1-e^{-\lambda x})^{\beta})}{1-\frac{2\,e^{a}}{e^{2a}-1}\sinh(a(1-e^{-\lambda\,x^{\beta}}))}. \quad (6)$$

The failure rate is a key notion in reliability and survival analysis for measuring the ageing process. Understanding the shape of the failure rate is important in reliability theory, risk analysis and other disciplines. The concepts of increasing, decreasing, bathtub shaped (first decreasing and then increasing) and upside-down bathtub shaped (first increasing and then decreasing) failure rates for univariate distributions have been found very useful in reliability theory. The classes of distributions having these ageing properties are designated as the IFR, DFR, BUT and UBT distributions, respectively. For the HCEE distribution hazard rate function can be decreasing, increasing, upside-down bathtub-shaped and constant. The behavior of the hazard function (6) is difficult to determine analytically. Fig 2 shows some samples of possible shapes of the hazard rate function in IFR, DFR and BUT cases for certain values of the vector(a,β,λ).

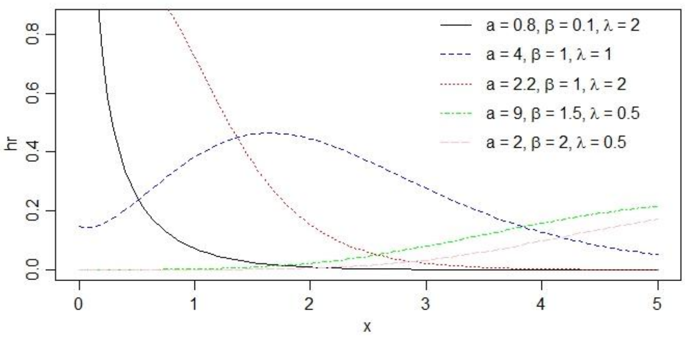

Figure 2. Failure rate function shapes for selected values of the parameters.

3.2. Moment generating function and mean residual life time

Now let us consider different moments of the $HCEE(a,\beta,\lambda)$ distribution. Some of the most important features and characteristics of a distribution can be studied through its moments, such as moment generating function, the jth moment and interested reliability properties such as mean residual life time. The moment generating function of $HCEE(a,\beta,\lambda)$ using (1) and (3) is immediately written as

$$M_X(t) = E(e^{tx}) = \frac{2\beta a\, e^a}{e^{2a}-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum\nolimits_{l=0}^{\infty}\frac{a^{2n}(-1)^{m+l}}{(2n)!}\binom{\beta-1}{m}\binom{2n\beta}{l}\frac{\lambda}{\lambda(m+l+1)-t}.$$

The jth moment of the HCEE distribution can be derived as

$$\mu_j = E(X^j) = \frac{2\beta a\, e^a}{e^{2a}-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum\nolimits_{l=0}^{\infty}\frac{a^{2n}(-1)^{m+l}}{(2n)!}\binom{\beta-1}{m}\binom{2n\beta}{l}\frac{\lambda\, j!}{(\lambda(m+l+1))^{j+1}}.$$

In particular, its mean and variance are given by

$$E(X) = \frac{2\beta a\, e^a}{e^{2a}-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum\nolimits_{l=0}^{\infty}\frac{a^{2n}(-1)^{m+l}}{(2n)!}\binom{\beta-1}{m}\binom{2n\beta}{l}\frac{\lambda}{(\lambda(m+l+1))^2}$$

And

$Var\,(X) = \mu_2 - {\mu_1}^2.$

One of the well-known properties of the life time distribution is mean residual life time. For the HCEE distribution it can be written as

$$m(t) = E(X-t|X>t) = \frac{2a\, e^a}{e^{2a}-1-2e^a\, sinh\ a\left(1-e^{-\lambda t^{\beta}}\right)}\times$$

$$\frac{2a\, e^a}{e^{2a}-1}\sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\sum\nolimits_{l=0}^{\infty}\frac{a^{2n}(-1)^{m+l}}{(2n)!}\binom{\beta-1}{m}\binom{2n\beta}{l}\frac{e^{-\lambda(m+l)t}}{\lambda(m+l)}.$$

3.3. Order statistics, stress-strength parameter and Shannon entropy measure

Here we provide some results about order statistics, stress-strength parameter and Shannon entropy of HCEE distribution. Let X_1 , X_2 ,… , X_n be a random sample from a $HCEE(a, \beta, \lambda)$, and let $X_{i:n}$denote the ith order statistic. The PDF of $X_{i:n}$ is given by

$$f_{X_{i:n}}(x) = \frac{n!}{(i-1)!\,(n-i)!}\frac{2a\, e^a\beta\lambda e^{-\lambda x}\left(1-e^{-\lambda x}\right)^{\beta-1}\cosh\left(a\left(1-e^{-\lambda x}\right)^{\beta}\right)}{e^{2a}-1}\times$$
$$\left(\frac{2\, e^a}{e^{2a}-1}sinh(a(1-e^{-\lambda x})^{\beta})\right)^{i-1}\left(1-\frac{2\, \mathrm{e}^{\mathrm{a}}}{\mathrm{e}^{2\mathrm{a}}-1}\mathrm{sinh}(\mathrm{a}(1-e^{-\lambda x})^{\beta})\right)^{n-i}.$$

Now we discuss about the stress-strength parameter. Suppose $X_1 \sim HCEE(a_1, \beta_1, \lambda_1)$ and $X_2 \sim HCEE(a_2, \beta_2, \lambda_2)$ are independently distributed, then

$$P(X_1 < X_2) = \frac{2\mathrm{a}_2 e^{\mathrm{a}_1+\mathrm{a}_2}}{(e^{2\mathrm{a}_1}-1)(e^{2\mathrm{a}_2}-1)}\times$$
$$\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}\sum_{l=0}^{\infty}\sum_{k=0}^{\infty}\sum_{j=0}^{\infty}\lambda_2\beta_2\frac{{\mathrm{a}_1}^{2n+1}\,{\mathrm{a}_2}^{2m}}{(2n+1)!\,(2m)!}\frac{(-1)^{k+l+j}\binom{\beta_1(2n+1)}{k}\binom{2\beta_2 m}{k}\binom{\beta_1-1}{j}}{(\lambda_1 l\beta_1(2n+1)+\lambda_2(2k\beta_2 m+j(\beta_1-1)+1))}.$$

The entropy of a random variable measures the variation of the uncertainty. A large value of entropy indicates the greater uncertainty in the data. Shannon entropy of $HCEE(a, \beta, \lambda)$Can be obtained as

$$H(X) = -log\left(\frac{2a\, e^a}{e^{2a}-1}\right) - \log(\beta\lambda) + \lambda E(X) - (\beta-1)E\left(\log\left(1-e^{-\lambda x}\right)\right) \quad - E\left(\log cosh\left(1-e^{-\lambda x}\right)^{\beta}\right).$$

4. Different Methods of Estimation

In this section, we describe the eight estimation methods considered in this paper for estimating the parameters a, β and λ of the HCEE distribution. For all methods we consider the case when a, β and λ are unknown.

4.1. Maximum likelihood estimation

Let $X_1, X_2, \dots, X_n$ be a random sample from the distribution with density $f(x, \underline{\theta})$. The likelihood function based on observed values $x_1, x_2, \dots, x_n$ is given by

$$L(\underline{\theta}, \underline{\boldsymbol{x}}) = \prod_{i=1}^{n} f(x_i, \underline{\theta})) \qquad (7)$$

By maximizing (7) the Maximum likelihood estimate of θ (MLE) is obtained. In case of the HCEE distribution, the log-likelihood function of the parameter is given as

$$l(a,\beta,\lambda,\underline{\boldsymbol{x}}) = log\left(L(a,\beta,\lambda,\underline{\boldsymbol{x}})\right)$$
$$= n\, ln\frac{2a\, e^{a}}{e^{2a}-1} + n\, ln\, \lambda\beta - \lambda\sum_{i=1}^{n} x_i + (\beta-1)\sum_{i=1}^{n} \ln\left(1-e^{-\lambda x_i}\right) + \sum_{i=1}^{n} ln\, cosh\, a\left(1-e^{-\lambda x_i}\right)^{\beta}.$$

So, the MLEs of a, β and λ, say $\hat{a}$, $\hat{\beta}$ and $\hat{\lambda}$, respectively, can be obtained as the solutions of

$$\frac{dl}{da} = n\left(\frac{a+1}{a} - \frac{2e^{2a}}{e^{2a}-1}\right) + \sum_{i=1}^{n}\frac{(1-e^{-\lambda x_i})^{\beta}\ sinh\, a(1-e^{-\lambda x_i})^{\beta}}{cosh\, a(1-e^{-\lambda x_i})^{\beta}} = 0,$$

$$\frac{dl}{d\beta} = \frac{n}{\beta} + \sum_{i=1}^{n} ln(1-e^{-\lambda x_i}) + \sum_{i=1}^{n}\frac{a(1-e^{-\lambda x_i})^{\beta} ln(1-e^{-\lambda x_i})\ sinh\, a(1-e^{-\lambda x_i})^{\beta}}{cosh\, a(1-e^{-\lambda x_i})^{\beta}} = 0,$$

$$\frac{dl}{d\lambda} = \frac{n}{\lambda} - \sum_{i=1}^{n} x_i + (\beta-1)\sum_{i=1}^{n}\frac{x_i e^{-\lambda x_i}}{1-e^{-\lambda x_i}} + \sum_{i=1}^{n}\frac{x_i a\beta\, e^{-\lambda x_i}(1-e^{-\lambda x_i})^{\beta-1}\ sinh\, a(1-e^{-\lambda x_i})^{\beta}}{cosh\, a(1-e^{-\lambda x_i})^{\beta}} = 0.$$

Due to the non-linearity of these equations the MLEs of parameters can be obtained numerically. We use the optim function from the statistical software R (R Development Core Team, 2011) to solve these equations.

4.2. Maximum product of spacings estimator

Maximum product of spacings (MPS) method was introduced by Cheng and Amin (1983) as an alternative to the MLE method. Ranneby (1984) derived the MPS method from an approximation of the Kullback-Leibler divergence (KLD). Kullback-Leibler divergence between $f(x,\underline{\theta})$ and $f(x,\ \underline{\theta}_0)$ is given by

$$KLD(f(x,\ \underline{\theta}_0)||f(x,\underline{\theta})) = \int f(x,\ \underline{\theta}_0)\log\left(\frac{f(x,\ \underline{\theta}_0)}{f(x,\underline{\theta})}\right)dx.$$

The KLD is zero if and only if $f(x,\underline{\theta}) = f(x,\ \underline{\theta}_0)$ for all$x$.

Let $x_1,\dots,x_n$ be a sample from a CDF $F(x,\theta)$. Let $f(x,\theta)$ denote the corresponding PDF. For estimating $\underline{\theta}$ a perfect method should make the KLD between the model and the true distribution as small as possible. In applications, this can be approximated by estimating

$$\frac{1}{n}\sum_{i=1}^{n}\log\left[\frac{f\left(x_i,\ \underline{\theta}_0\right)}{f(x_i,\underline{\theta})}\right]. \tag{8}$$

So, by minimizing (8) with respect to$\underline{\theta}$, the estimator of $\underline{\theta}_0$ can be found. Ranneby (1984) suggested another approximation of the KLD, namely

$$\frac{1}{n}\sum_{i=1}^{n+1}\log\left[\frac{F\left(x_{(i)},\ \underline{\theta}_0\right) - F\left(x_{(i-1)},\underline{\theta}_0\right)}{F\left(x_{(i)},\underline{\theta}\right) - F\left(x_{(i-1)},\underline{\theta}\right)}\right], \tag{9}$$

Where $x_{(i)}, i = 1, 2, \dots, n$ denotes the ordered sample and $F\left(x_{(0)}\right) = 0,\ F\left(x_{(n+1)}\right) = 0$.

The estimator obtained by minimizing (9) is called the MPS estimator of θ_0. It is clear that minimizing (9) is equivalent to maximizing

$$\sum_{i=1}^{n+1}\log[F\left(x_{(i)},\underline{\theta}\right) - F\left(x_{(i-1)},\underline{\theta}\right)].$$

In case of the HCEE distribution, the MPSs of a,β and λ, say $\hat{a}_{MPS},\hat{\beta}_{MPS}$ and $\hat{\lambda}_{MPS}$, respectively, can be obtained by minimizing

$$\sum_{i=1}^{n+1}\log[G\left(x_{(i)},a,\beta,\lambda\right) - G\left(x_{(i-1)},a,\beta,\lambda\right)] =$$
$$\sum_{i=1}^{n+1}\log[\frac{2\,e^{a}}{e^{2a}-1}\sinh(a\left(1-e^{-\lambda x_{(i)}}\right)^{\beta}) - \frac{2\,e^{a}}{e^{2a}-1}\sinh(a\left(1-e^{-\lambda x_{(i-1)}}\right)^{\beta})].$$

With respect to a,β and λ.

4.3. Estimators based on percentiles

Estimation based on percentiles was originally explored by Kao (1958, 1959). In fact the nature of percentiles estimators is based on distribution function can be obtained by minimizing

$$\sum_{i=1}^{n}\left[x_{(i)} - F^{-1}(p_i,\theta)\right]^2,$$

Where $p_i = \frac{i}{n+1}$ and $x_{(i)}$; $i = 1, \dots, n$ denotes the ordered sample. So to obtain the PC estimator of a and λ, we use the same method as for the ML estimator. In case of the HCEE distribution, the PCEs of a, β and λ, say $\hat{a}_{PCE}, \hat{\beta}_{PCE}$ and $\hat{\lambda}_{PCE}$, respectively, can be obtained by minimizing

$$\sum_{i=1}^{n} [x_{(i)} - G^{-1}(p_i, a, \beta, \lambda)]^2 =$$

$$\sum_{i=1}^{n} [x_{(i)} + \frac{1}{\lambda} log(1 - (\frac{\log(\frac{e^{2a}-1}{2e^a}(\frac{i}{n+1}) + \sqrt{\left(\frac{e^{2a}-1}{2e^a}(\frac{i}{n+1})\right)^2 + 1})}{a})^{\frac{1}{\beta}})]^2$$

with respect to a, β and λ.

4.4. Least squares and weighted least squares estimators

In this section, we derive regression based estimators of the unknown parameter. This method was originally suggested by Swain et al. (1988) to estimate the parameters of beta distributions. It can be used for some other distributions also. Suppose $X_1, X_2, \dots, X_n$ is a random sample of size n from a CDF $F(\cdot)$ and suppose $X_{(j}, i = 1, 2, \dots, n$ denote the ordered sample in ascending order. The proposed method uses $F(X_{(j)})$. For a sample of size n, we have

$$E[F(X_{(j)})] = \frac{j}{n+1}, \quad Var[F(X_{(j)})] = \frac{j(n-j+1)}{(n+1)^2(n+2)},$$

$$Cov[F(X_{(j)}), F(X_{(l)})] = \frac{j(n-l+1)}{(n+1)^2(n+2)}, \quad j < l.$$

Using the expectations and the variances, two variants of the least squares method follow.

Method 1: Least squares estimators

The least squares estimators can be obtained by minimizing

$$\sum_{j=1}^{n} \left[F(x_{(j)}) - \frac{j}{n+1}\right]^2$$

with respect to the unknown parameters. In case of the HCEE distribution, the LSEs of a, β and λ, say $\hat{a}_{LSE}, \hat{\beta}_{LSE}$ and $\hat{\lambda}_{LSE}$, respectively, can be obtained by minimizing

$$\sum_{j=1}^{n} [G(x_{(j)}, a, \beta, \lambda) - \frac{j}{n+1}]^2 =$$

$$\sum_{j=1}^{n} \left[\frac{2e^a}{e^{2a}-1} \sinh(a(1 - e^{-\lambda x_{(j)}})^\beta) - \frac{j}{n+1}\right]^2$$

with respect to a, β and λ.

Method 2: Weighted least squares estimators

The weighted least squares estimators can be obtained by minimizing

$$\sum_{j=1}^{n} w_j \left[F(x_{(j)}) - \frac{j}{n+1}\right]^2$$

with respect to the unknown parameters, where

$$w_j = \frac{1}{Var[F(X_{(j)})]} = \frac{(n+1)^2(n+2)}{j(n-j+1)}$$

In case of the HCEE distribution, the WLSEs of a, β and λ, say $\hat{a}_{WLSE}, \hat{\beta}_{WLSE}$ and $\hat{\lambda}_{WLSE}$, respectively, can be obtained by minimizing

$$\sum_{j=1}^{n} w_j [G(x_{(j)}, a, \beta, \lambda) - \frac{j}{n+1}]^2 =$$

$$\sum_{j=1}^{n} w_j [\frac{2e^a}{e^{2a}-1} \sinh(a(1 - e^{-\lambda x_{(j)}})^\beta) - \frac{j}{n+1}]^2$$

with respect to a, β and λ.

4.5. Bootstrap estimator (bootstrap confidence intervals)

The uncertainty in the parameters of the fitted distribution can be estimated by parametric (resampling from the fitted distribution) or nonparametric (resampling with replacement from the original data set) bootstraps resampling Efron and Tibshirani (1994). These two parametric and nonparametric bootstrap procedures are described as follows.

Parametric bootstrap procedure

1. Estimate θ (vector of unknown parameters), say $\hat{\theta}$, from sample on the MLE procedure.
2. Generate a bootstrap sample $\{X_1^*, \dots, X_m^*\}$, using $\hat{\theta}$. Obtain the bootstrap estimate of θ, say $\hat{\theta}^*$, from the bootstrap sample based on the MLE procedure.
3. Repeat Step2 NBOOT times.
4. Order $\hat{\theta}^*{}_1, \dots, \hat{\theta}^*{}_{NBOOT}$ as $\hat{\theta}^*{}_{(1)}, \dots, \hat{\theta}^*{}_{(NBOOT)}$. Then obtain γ −quantiles and $100(1-\gamma)\%$ confidence intervals of parameters.

In case of the HCEE distribution, the parametric bootstrap estimators (PBs) of a, β and λ, say $\hat{a}_{PB}, \hat{\beta}_{PB}$ and $\hat{\lambda}_{PB}$, respectively.

Nonparametric bootstrap procedure

1. Generate a bootstrap sample $\{X_1^*, \dots, X_m^*\}$ with replacment from original data set. Obtain the bootstrap estimate of θ with MLE procedure, say $\hat{\theta}^*$ using the bootstrap sample.
2. Repeat Step2 NBOOT times.
3. Order $\hat{\theta}^*{}_1, \dots, \hat{\theta}^*{}_{NBOOT}$ as $\hat{\theta}^*{}_{(1)}, \dots, \hat{\theta}^*{}_{(NBOOT)}$.Then obtain γ −quantiles and $100(1-\gamma)\%$ confidence intervals of parameters.

In case of the HCEE distribution, the nonparametric bootstrap estimators (NPBs) of a, β and , say $\hat{a}_{NPB}, \hat{\beta}_{NPB}$ and $\hat{\lambda}_{NPB}$, respectively.

4.6. Bayesian estimation

In this section, we have a short note on the Bayes estimation of the parameters of HCEE distribution.To do this, assume that the vector of unknown parameters (a, β, λ) have independent prior distributions. Then, by attention to $a > 0, \beta > 0$ and $\lambda >$, we consider

$a \sim Gamma(b, c), \beta \sim Gamma(d, e)$ and $\lambda \sim Gamma(f, g)$

where all of b, c, d, e, f and g are positive parameters. Then, the joint posterior probability density function of α and λ given $\underline{x} = (x_1, x_2, \dots, x_n)$ can be written as:

$\pi^*(a, \beta, \lambda | \underline{x}) \propto \pi(a, \beta, \lambda) f(\underline{x}, a, \beta, \lambda)$

where $\pi(\alpha, \beta, \lambda)$ is the joint prior distribution of the parameters. Since this posterior distribution is cumbersome, we can not provide posterior estimates of the parameters theoretically, but, by using MCMC algorithm in WINBUGS software we will obtain this estimators. The Bayesian estimators **(**Bs) of a, β and λ, say $\hat{a}_B, \hat{\beta}_B$ and $\hat{\lambda}_B$, respectively.

5. Simulations

In this section, we perform a simulation study to investigate the finite sample properties of ML estimators described in Section 4. To conduct the experimental study, we generate 5000 synthetic samples of size n = 10, 30, 50, 100 and 200 from HCEE with true selected parameters $C_1 = (a = 1, \beta = 2, \lambda = 2)$ and $C_2 = (a = 2, \beta = 2, \lambda = 2)$. To examine the estimation accuracies, the absolute bias and the mean squared error (MSE) are computed**.** Figures 3 -6 show a graphical representation of the absolute bias and the MSE of the parameter estimates as a function of sample size n.

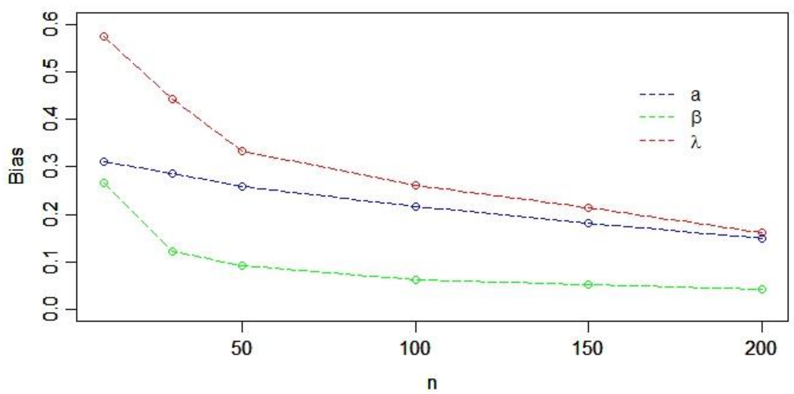

Figure 3. Absolute bias of selected parameters $(a = 1, \beta = 2, \lambda = 2)$ for HCEE model.

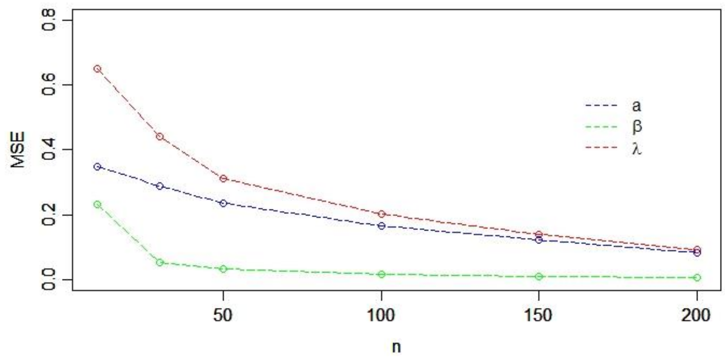

Figure 4. Absolute MSE's of selected parameters$(a = 1, \beta = 2, \lambda = 2)$ for HCEE model.

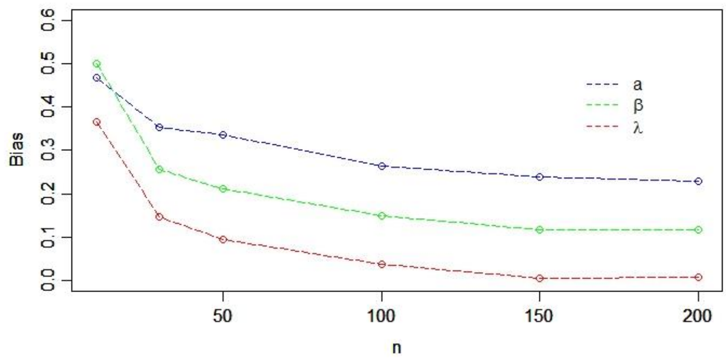

Figure 5. Absolute bias of selected parameters $(a = 2, \beta = 2, \lambda = 2)$ for HCEE model.

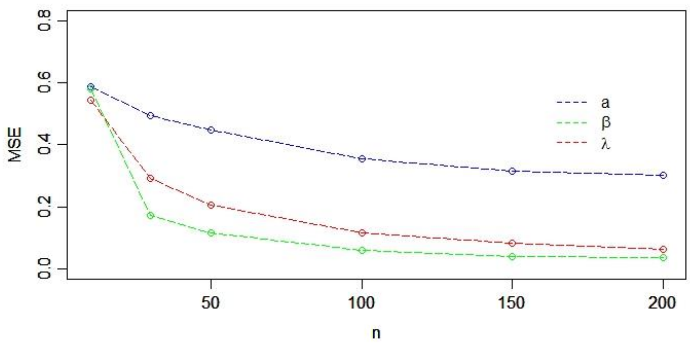

Figure 6. Absolute MSE's of selected parameters$(a = 2, \beta = 2, \lambda = 2)$ for HCEE model.

Clearly, the bias and MSE of three parameters converge to zero when n increases.

6. Data analysis and application

In this section, we illustrate the usefulness of the HCEE distribution. First, the parameters of HCEE distribution are estimated for the yarn data set by eight estimation methods. Second, we fit the HCEE model to this data by ML method and compare the results with the gamma ,Weibull and generalized exponential (GE) with respective densities

$$f_{gamma}(x) = \frac{1}{\Gamma(\alpha)} \lambda^{\alpha} x^{\alpha-1} e^{-\lambda x}, \quad x \geq 0$$

$$f_{Weibull}(x) = \frac{\beta}{\lambda^\beta} x^{\beta-1} e^{-(\frac{x}{\lambda})^\beta}, \qquad x \geq 0,$$

$$f_{GE}(x) = \alpha\lambda e^{-\lambda x}(1 - e^{-\lambda x})^{\alpha-1}, \quad x \geq 0\,.$$

Yarn data

The data set relates to the time-to-failure of apolyester/viscose yarn in a textile experiment for testing the tensile fatigue characteristics of yarn. It consists of a sample of 100 cm yarn at 2.3% strain level (see Oluyede et al., 2016). The data are given below:

86 146 251 653 98 249 400 292 131 169 175 176 76 198 180 321 42 220 157 264 88 262 195 364 15 264 229 90 196 250 325 180 149 224 282 121 61 20 38 180 211 246 597 135 40 40 341 151 65 337 38 166 229 185 423 182 497 186 81 279 124 571 353 315 93 244 61 55 55 568 188 185 246 71 398 290 338 400 198 143 277 194 286 236 239 829 203 396 393 284 20 264 105 203 124 137 135 350 193 188**.**

Before analyzing this data set, we use the scaled-TTT plot to verifiy our model validity, see Aarset (1987). It allows to identify the shape of hazard function graphically. We provide the empirical scaled-TTT plot for the yarn data. Fig. 7 shows the scaled-TTT plots is concave. It indicates that the hazard function is increasing; therefore it verifies our model validity.

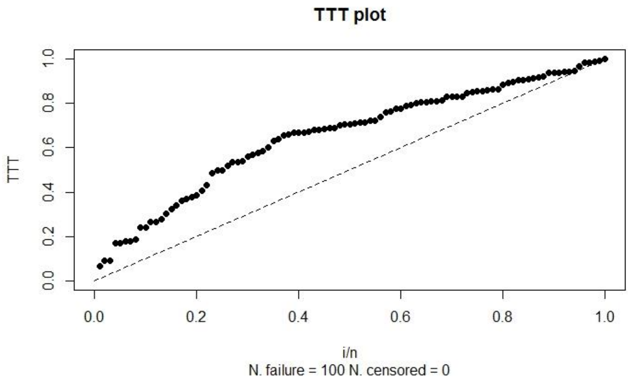

Figure 7. Scaled-TTT plot for the yarn data.

Now we apply different estimation methods for the parameters of HCEE distribution. Also, the performances of estimators can be compared through log-likelihood function. Table 1 shows the estimation of parameters of HCEE distribution and corresponding log-likelihood function for the yarn data that obtained by eight estimation methods: maximum likelihood, Bayesian, maximum product of spacings, parametric bootstrap, non-parametric bootstrap, percentile, least-squares and weighted least-squares.

Table 1. Estimates of the parameters and the corresponding log-likelihood for the yarn data.

Method	Estimate of α	Estimate of β	Estimate of $\boldsymbol{\lambda}$	Log-likelihood
MLE	2.490	1.879	0.009	-624.566
LSE	3.120	1.593	0.0097	-624.198
WLSE	3.122	1.484	0.0094	-624.206
PCE	2.469	1.624	0.0086	-624.325
MPS	3.030	1.435	0.0086	-624.723
PB	2.395	1.946	0.0092	-623.965
NPB	2.458	1.931	0.0093	-623.917
Bayes estimation under quadratic loss function				
B	2.357	1.901	0.0102	-625.451
Bayes estimation under absolute loss function				
B	2.883	1.823	0.0102	- 624.343

Here, we fit the HCEE distribution to the yarn data and compare it with the gamma, generalized exponential and Weibull densities. Table 2 shows the MLEs of parameters, log-likelihood, Akaike information criterion (AIC), Cramér–von Mises (W^*) and Anderson–Darling (A^*) statistics for the yarn data. The selection criterion is that the lowest AIC, A^* and W^*correspond to the best fit model. Thus, the HCEE distribution provides the best fit for the yarn data set as it shows the lowest AIC, A^* and W^* than other considered models. The relative histograms, fitted HCEE, gamma, generalized exponential and Weibull PDFs for the yarn data are plotted in Fig 8(a). The plots of empirical and fitted survival functions, P-P plots and Q-Q plots for the HCEE and other fitted distributions are displayed in Fig 8(b), Fig 8(c) and Fig 8(d), respectively. These plots also support the results in Table 2.

Table 2. The MLEs of parameters for the yarn data.

Model	MLEs of parameters	Log-likelihood	AIC	A*	W*
HCEE	$\boldsymbol{\hat{a} = 2.490,\ \hat{\lambda} = 0.009, \hat{\beta} = 1.879}$	**-623.901**	**1253.802**	**0.337**	**0.052**
gamma	$\hat{\alpha} = 2.238,\ \hat{\lambda} = 0.316,$ ---	-625.244	1254.489	0.667	0.123
Weibull	$\hat{\beta} = 1.604,\ \hat{\lambda} = 247.903,$ ---	-625.199	1254.398	0.529	0.092
GE	$\hat{\alpha} = 2.384,\ \hat{\lambda} = 0.007,$ ---	-625.693	1255.386	0.529	0.152

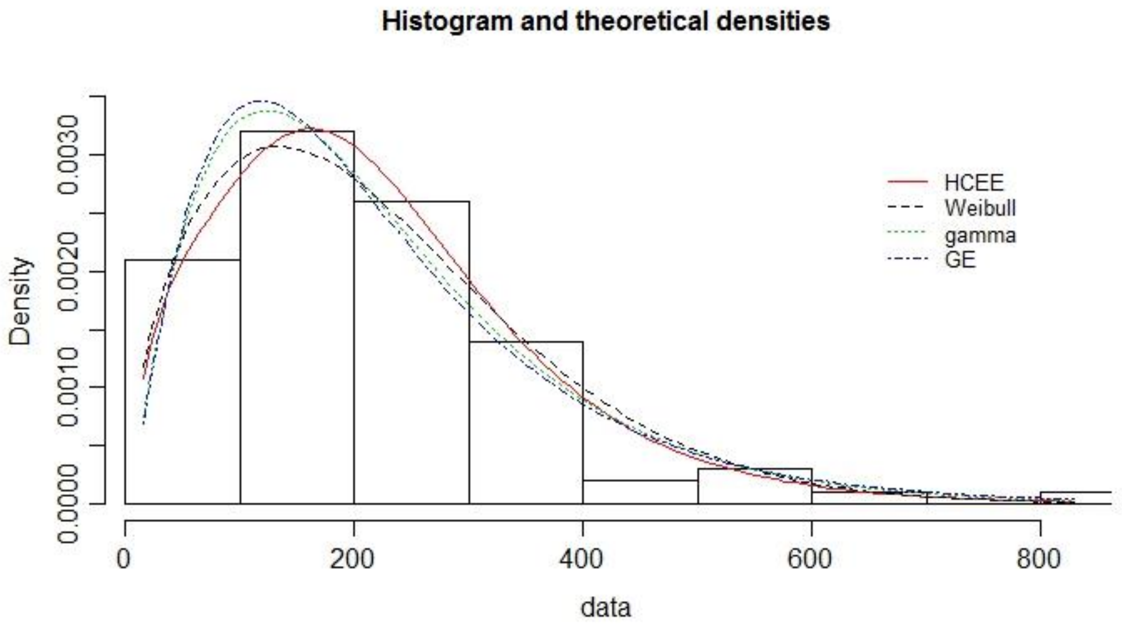

Figure 8(a). The fitted PDFs and the relative histogram for the yarn data set.

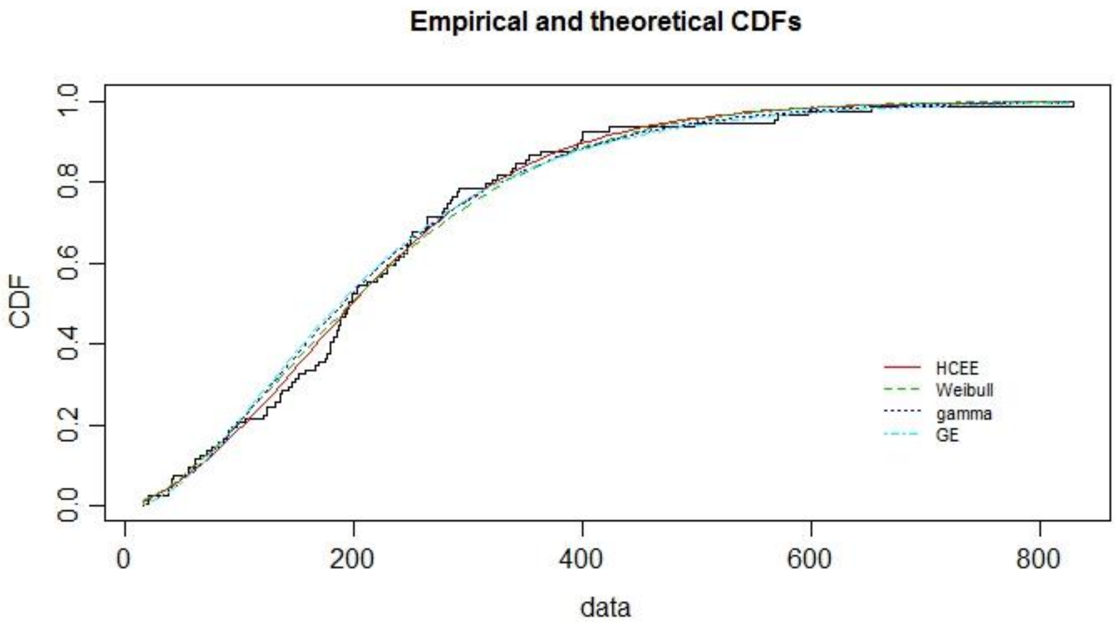

Figure 8(b). Empirical and fitted survival functions for the yarn data set.

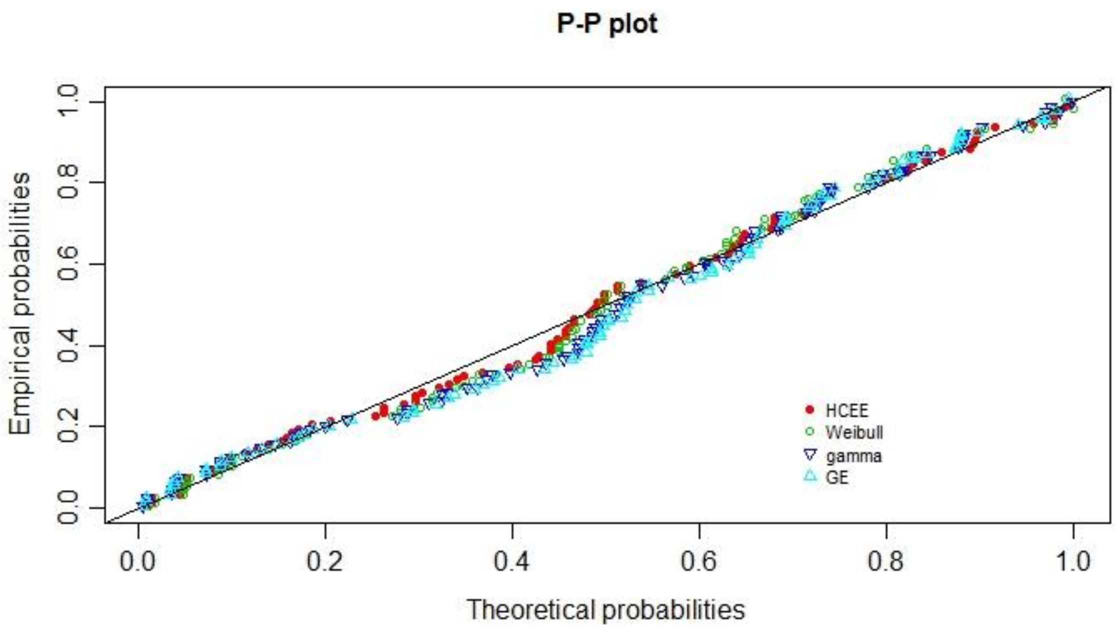

Figure 8(c). P-P plots of fitted PDFs for the yarn data set.

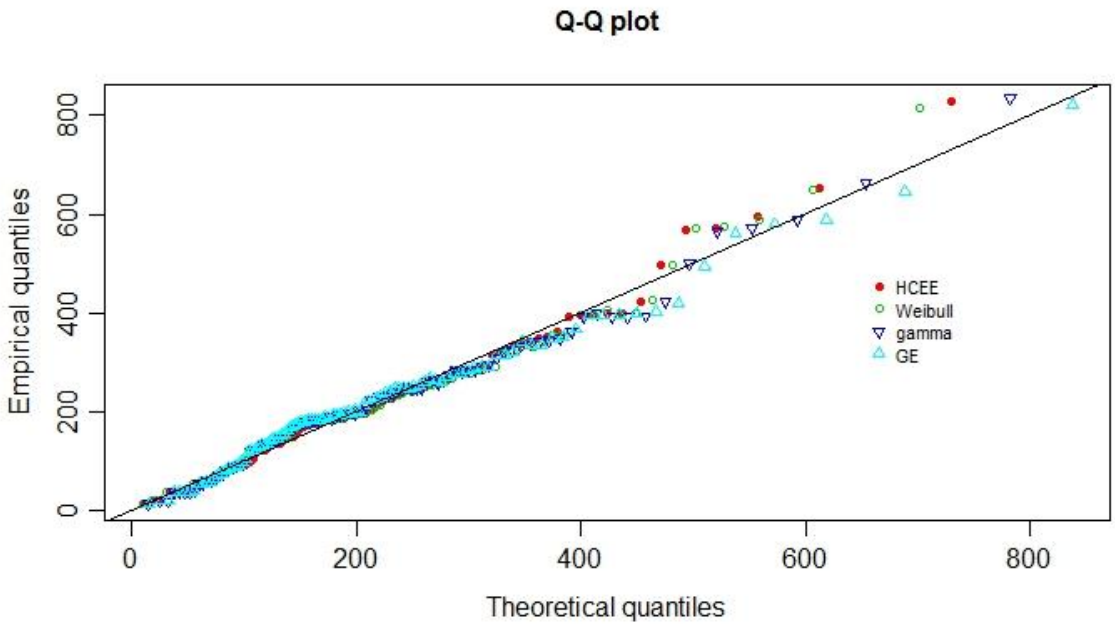

Figure 8(d). Q-Q plots of fitted PDFs for the yarn data set.

7. Conclusions

We introduce a new model, so-called the HCEE distribution that is a special cases of HCF family of distributions proposed by Kharazmi and Saadatinik (2016). It is expected that this family will be widely applicable in reliability theory, risk analysis and other disciplines. The HCEE distribution, is very strong competitor to other well-known distributions commonly used in literature for fitting statistical data. Estimation of parameters is approached by the methods of maximum likelihood, Bayesian, maximum product of spacings, parametric bootstrap, non-parametric bootstrap, percentile, least-squares and weighted least-squares. A simulation study is conducted to examine the bias, mean square error of the maximum likelihood estimators. Moreover, one application of the HCEE distribution to real data set is provided to illustrate that this distribution provides a better fit than Weibull, gamma and generalized exponential distributions.

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