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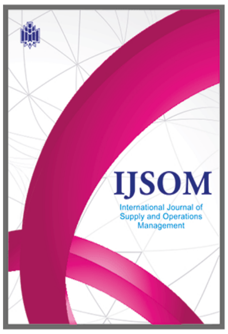

EOQ model for delayed deteriorating items with shortages and trade credit policy

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Abstract
This paper deals with a deterministic inventory model for deteriorating items under the condition of permissible delay in payments with constant demand rate which differs from before and after deterioration for a single item. Shortages are allowed and completely backlogged. Under these assumptions, this paper develops a retailer's model for obtaining an optimal cycle length and ordering quantity in deteriorating items of an inventory model. Thus, our objective is retailer's cost minimization problem to find an optimal replenishment policy under various parameters. The convexity of the objective function is derived and the numerical examples are provided to support the proposed model. Sensitivity analysis of the optimal solution with respect to major parameters of the model is included and the implications are discussed.

Keywords: Inventory; Deterioration; Constant demand rate; Trade credit; Complete backlogging.

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1. Introduction

In the classical Economic Ordering Quantity (EOQ) model, two basic questions should be answered and that are when and how much to order when any inventory control model is concerned. The main assumption in an EOQ model is that shortages are not permitted. But this assumption is unrealistic when a real-life situation is concerned. Also, the customers are willing to wait for delivery of goods; planned back orders can even make sense. In a deterministic inventory model demand rate is a function of time is known, which implies that orders will be placed when the stock in an inventory is abundant enough to meet the demand.

The decaying inventory problem was first addressed by Ghare and Shrader (1963) who developed EOQ model with a constant rate of decay, the same work was extended by Covert and Philip (1973) for a variable rate of deterioration. Shah (1977) generalized the work of Ghare and Shrader (1963) by allowing backordering. Hollier and Mark (1983). Heng et al. (1991), Datta and Pal (1990) and Goswami and Chaudari (1991) who worked on deteriorating items without considering delay in payments. Papachristos and Skouri (2000), Chang and Dye (1999) who looked at deteriorating items in a continuous review environment with shortages while Cheng and Chen (2004) considered deteriorating items in a periodic review environment with shortages. Maya Gayen and Pal A.K. (2009) also examined deteriorating items in two-warehouse problem.

Condition of permissible delay in payment was not considered prior to 1985 when Goyal (1985) developed an EOQ model under such condition. Aggarwal and Jaggi (1995) extended Goyal’s model for deteriorating items. Jamal et al. (1997) extended the work of Aggarwal and Jaggi by allowing for shortages. Meddh et al. (2004) investigated the effect of permissible delay in a periodic review situation, Ouyang et al. (2005) who added the condition of cash discount and Teng (2002) who considered the difference between unit price and cost of items. Shin et al. (1996), Jamal et al. (2000) and Liao et al. (2000) who looked at delay in payments under inflation. Permissible delay in payments under continuous review environment was looked by Salameh et al. (2003) while Chung and Huang (2003) looked at the situation under the EPQ production model. Huang (2003), Chung et al. (2005) and Teng (2002), Thangam and Uthayakumar (2010) all developed models under permissible delay in payments.

In our case, we extend Abubakar et al. (2012) work by allowing shortages which are completely backlogged. Ouyang et al. (2006) and Chung (2009) who developed an inventory model in which the demand rate before and after the deterioration sets is a constant as in the case of authors mentioned earlier. The (Net Present Value) NPV approach is considered as the right approach see for instance Grubbstrom (1980), Van der Lann (2003). In reverse logistics inventory models the average annual cost approach could be inappropriate to use Teunter and Van der Lann (2002). The average annual cost method is good approximation to the NPV method for EOQ models. For parameter values may be encountered by Hadley (1964) and for non-slow-moving items Klein and Teunter (1998).

In this study, we develop an inventory model for deteriorating items under the condition of permissible delay in payment by allowing shortages which are completely backlogged and which is realistic in real-time situation. The remainder of this paper is organized as notations and assumptions in Section 2. In Section 3, the mathematical model is developed to minimize the total cost followed by computation of the total inventory cost in Section 4. In Section 5, analysis and optimization of the model is discussed. In order to execute our present model an algorithm is provided which is due to Yanlai Liang and Fangming Zhou (2011) in Section 5.1. Several numerical examples are provided in section 6, followed by sensitivity analysis in Section 7.The paper closes with concluding remarks in section 8.

2. Notations and assumptions

The following notations and assumptions are used in this paper.

2.1. Notations

D_1 demand rate (units per unit time) during the period before the deterioration sets in

D_2 demand rate (units per unit time) during the period after the deterioration sets in

Q the retailer's order quantity

t_d time the deterioration sets in

t_1 time at which the inventory level falls to zero

T inventory cycle length

T_2 time difference between the inventory cycle length and the time the deterioration sets in

T_1 time difference between shortage time and the deterioration sets in

I_d inventory level at the time the deterioration sets in

c unit purchasing cost

s unit selling price

k ordering cost per order

h unit stock holding cost per unit of time (excluding interest charges)

p interest payable per cycle

I_c interest charges per $ in stocks per unit of time by the supplier

I_e interest, which can be earned per $ per unit of time by the retailer

M the retailer's trade credit period offered by supplier in years

θ rate of deterioration

E_1 interest earned from the accrued sales

$d(T_2)$ number of items that deteriorate during the time interval $[T_1, T_2]$

δ the backlogging fraction

c_2 shortage cost per unit per order

2.2. Assumptions

1. Replenishment is instantaneous
2. Lead time is negligible
3. Shortages are allowed to occur. Only a fraction δ of it is backlogged

3. Mathematical model

Let $I(T)$ be the inventory level at any time t, $(0 \leq t \leq T)$. During the time interval $[0, t_d]$, the inventory level is decreasing only owing to stock-dependent rate. The inventory level is dropping to zero due to demand and deterioration during the time interval $[t_d, t_1]$. Then shortage interval keeps to the end of the current order cycle T. The whole process is repeated. Therefore the inventory system at any time t can be represented by the following differential equations:

$$I'(t) = \begin{cases} -D_1 & 0 \leq t < t_d \\ -D_2 - \theta I(t) & t_d \leq t < t_1 \\ -D_2 \delta & t_1 \leq t < T \end{cases} \tag{1}$$

with boundary conditions $I(t_d^-) = I(t_d^+)$ and $I(t_1) = 0$.

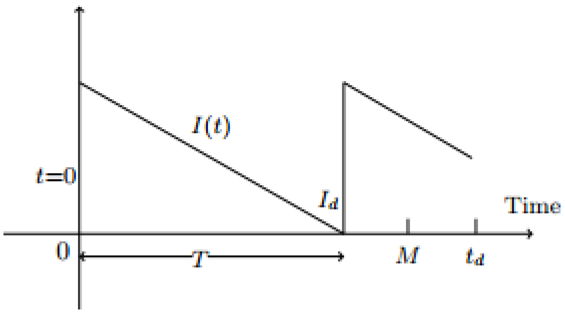

Figure 1: Inventory level over time in case 1.1: $T \leq M \leq t_d$

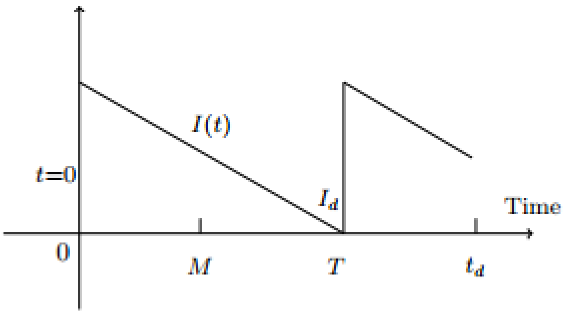

Figure 2: Inventory level over time in case 1.2: $M < T \leq t_d$

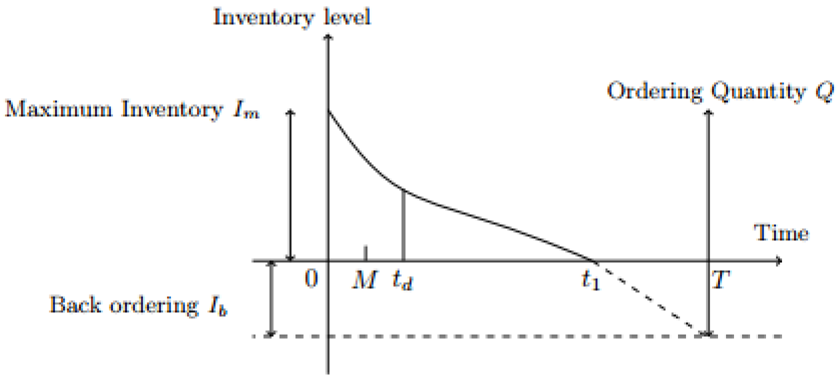

Figure 3: Inventory level over time in case 1.3: $M \leq t_d < T$

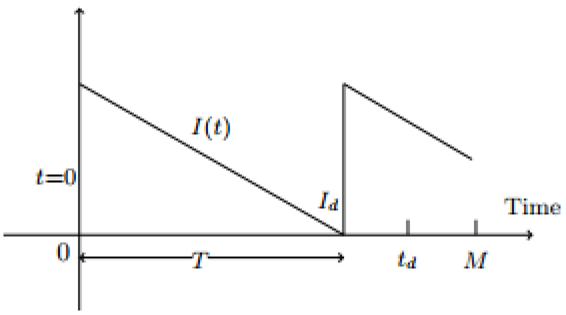

Figure 4: Inventory level over time in case 2.1: $T \leq t_d < M$

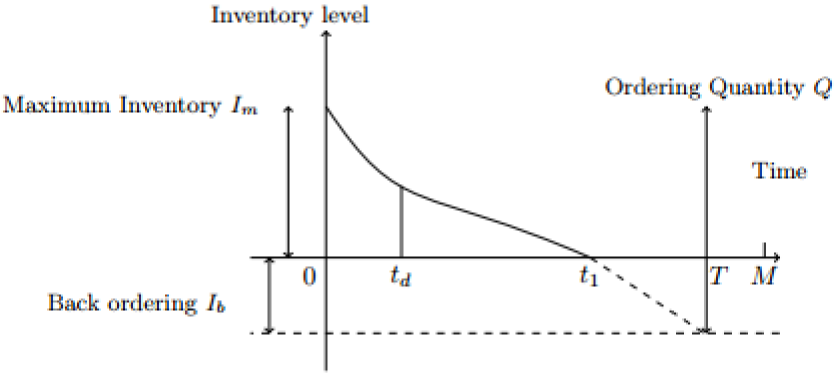

Figure 5: Inventory level over time in case 2.2: $t_d < T \leq M$

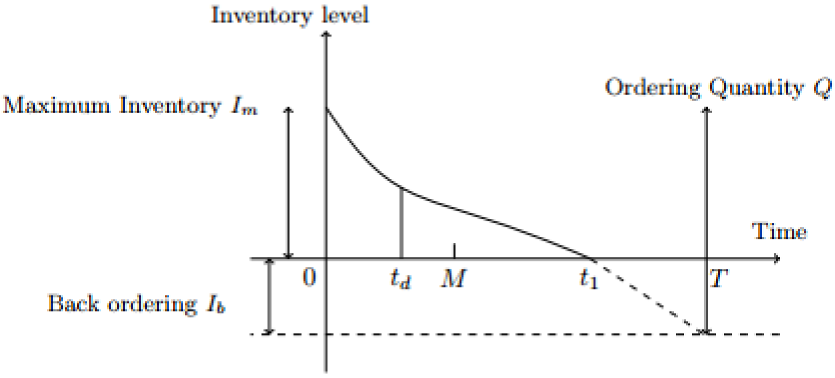

Figure 6: Inventory level over time in case 2.3: $t_d < M < T$

The solutions of the above differential equations after applying the boundary conditions are

$$I'(t) = \begin{cases} D_1(t_d - t) + \frac{D_2}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) & 0 \le t < t_d \\ \frac{D_2}{\theta}\left(e^{\theta(t_1 - t)} - 1\right) & t_d \le t < t_1 \\ -D_2\delta(t - t_1) & t_1 \le t < T \end{cases} \tag{2}$$

The maximum inventory level I_m is given by

$$I_m = D_1 t_d + \frac{D_2}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right)$$

The maximum amount of demand backlogged I_b is given by

$$I_b = D_2\delta(T - t_1)$$

The retailer's ordering quantity Q per cycle is given by

$$Q = I_m + I_b = I_m = D_1 t_d + \frac{D_2}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) + I_b = D_2\delta(T - t_1) \tag{3}$$

From the above solutions, inventory level at the time, the deterioration begins is given by

$$I_d = \frac{D_2}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right) \tag{4}$$

Thus, the number of items that deteriorate during $[t_d, T]$ is given by

$$d(T_2) = I_d - D_2 T_2\left(\frac{D_2}{\theta}\left(e^{\theta(t_1 - t_d)} - 1\right)\right) \tag{5}$$

3.1 Inventory scenarios

Depending upon whether $T \le t_d$ or $T > t_d$ there are two cases to be considered. Each of which is further subdivided into three sub-cases as follows:

Case 1. $M \le t_d$ with the sub-ases:

Case 1.1: $T \le t_d$

Case 1.2: $0 \le M < T \le t_d$

Case 1.3: $0 \le M \le t_d < T$

Case 2. $M > t_d$ with the sub-cases:

Case 2.1: $T \le t_d < M$

Case 2.2: $t_d < T < T \le M$

Case 2.3: $t_d < M < T$

3.2 Inventory carrying cost

Let the inventory holding cost per cycle be C_H, then for $M \le t_d$ the carrying cost is given by

$$C_H = ch\int_0^T D_1(T-t)dt = \frac{chD_1T^2}{2} \tag{6}$$

for **case 1.1** and **case 1.2** if $T \le t_d$.Otherwise

$$\begin{aligned} C_H &= ch\int_0^{t_d} I(t)\ dt + ch\int_{t_d}^{T} I(t)\ dt \\ &= ch\left\{\frac{D_1t_d^2}{2} + \frac{D_2t_d}{\theta}e^{\theta(t_1-t_d)} - \frac{D_2}{\theta}(\frac{1}{\theta}+1) + \frac{D_2}{\theta^2}e^{\theta(t_1-t_d)}\right\} \end{aligned} \tag{7}$$

for **case 1.3** if $t_d < T$.

3.3 Interest payable per cycle

For **case 1.1**,**case 2.1** and **case 2.2**, the interest payable P is zero since $M \ge T$. For **case 1.2**, **case 1.3** and **case 2.3** respectively, the interest payable P is given by

$$\begin{aligned} P &= CI_c \int_M^T I(t)\, dt \\ &= CI_c \left\{ D_1(t_d T - \frac{T^2}{2}) + \frac{D_2}{\theta} T(e^{\theta(t_1 - t_d)} - 1) - D_1(t_d M - \frac{M^2}{2}) \right. \\ &\quad \left. - \frac{D_2}{\theta} M(e^{\theta(t_1 - t_d)} - 1) \right\} \end{aligned} \tag{8}$$

$$-\frac{D_2}{\theta} M(e^{\theta(t_1 - t_d)} - 1) + \frac{D_2}{\theta}(-t_1 - \frac{1}{\theta} + \frac{e^{\theta(t_1 - t_d)}}{\theta} + t_d) \Big\} \tag{9}$$

$$P = CI_c \int_M^{t_1} I(t)\, dt = CI_c \frac{D_2}{\theta} \left\{ \frac{-1}{\theta} - t_1 + \frac{e^{\theta(t_1 - M)}}{\theta} + M \right\} \tag{10}$$

3.4. Interest earned per cycle

The interest earned from accrued sales, *E* is the interest earned during positive stock of the inventory. It is calculated based on the assumption that interest is earned by the customer up to the period allowed to settle the account, beyond this only, the interest is charged. Thus interest earned *E* is given for all the six cases as follows:

$$E = sI_e \left\{ \int_0^T D_1 t \, dt + D_1 T(M-T) \right\} = sI_e \left\{ \frac{D_1}{2} T^2 + D_1 T(M-T) \right\}$$

$$E = sI_e \int_0^M D_1 t \, dt = sI_e \frac{D_1}{2} M^2$$

$$E = sI_e \int_0^M D_1 t \, dt = sI_e \frac{D_1}{2} M^2$$

$$E = sI_e \left\{ \int_0^T D_1 t \, dt + D_1 T(t_d - T) + D_1 T(M - t_d) \right\}$$

$$= sI_e \left\{ \frac{D_1}{2} T^2 + D_1 T(t_d - T) + D_1 T(M - t_d) \right\}$$

$$E = sI_e \left\{ \int_0^{t_d} D_1 t \, dt + \int_{t_d}^T D_2 t \, dt + (D_1 t_d + D_2 (T - t_d))(M - T) \right\}$$

$$= sI_e \left\{ \frac{D_1}{2} t_d^2 + \frac{D_2}{2} (T^2 - t_d^2) + (D_1 t_d + D_2 (T - t_d))(M - T) \right\}$$

$$E = sI_e \left\{ \int_0^{t_d} D_1 t \, dt + \int_{t_d}^M D_2 t \, dt \right\} = sI_e \left\{ \frac{D_1}{2} t_d^2 + \frac{D_2}{2} (M^2 - t_d^2) \right\} \quad (11)$$

3.5. Shortage cost

Shortage cost C_s is given by

$$C_s = c_2 \int_{t_1}^T -I(t)dt = c_2 \int_{t_1}^T D_2 \delta (t - t_1) dt = \frac{c_2 D_2 \delta}{2} (T - t_1)^2 \quad (12)$$

4. Computation of the total inventory cost

Case 1. $M \leq t_d$

To determine the total inventory cost per unit time, the following notation is used:

$$TC_1(t_1, T) = \begin{cases} TC_{1,1}(t_1, T) & T \leq M < t_d \\ TC_{1,2}(t_1, T) & M \leq T < t_d \\ TC_{1,3}(t_1, T) & t_1 \leq t < T \end{cases} \quad (13)$$

Where

$$\begin{aligned} TC_{1,1}(t_1,T) &= \frac{1}{T}(k+C_H+P-E+C_s) \\ &= \frac{k}{T}+cD_1h\frac{T}{2}+sI_eD_1\frac{T}{2}-sI_eD_1M \\ &+ c_2D_2\delta(\frac{t}{2}-t_1+\frac{t_1^2}{2T}) \end{aligned} \tag{14}$$

$$\begin{aligned} TC_{1,2}(t_1,T) &= \frac{1}{T}(k+C_H+P-E+C_s) \\ &= \frac{k}{T}+cD_1h\frac{T}{2}+cI_c\left\{D_1(t_d-\frac{T}{2})+\frac{D_2}{\theta}(e^{\theta(t_1-t_d)}-1)\right. \\ &- \left.\frac{D_1}{T}(t_dM-\frac{M^2}{2})-\frac{D_2M}{\theta T}(e^{\theta(t_1-t_d)}-1)\right\} \\ &- sI_eD_1\frac{M^2}{2T}+c_2D_2\delta(\frac{T}{2}-t_1+\frac{t_1^2}{2T}) \end{aligned} \tag{15}$$

$$\begin{aligned} TC_{1,3}(t_1,T) &= \frac{1}{T}(k+cd(T_2)+C_H+P-E+C_s) \\ &= \frac{k}{T}+\frac{ch}{T}\left\{\frac{D_1}{2}T^2+\frac{D_2}{\theta}t_d(e^{\theta(t_1-t_d)}-1)-\frac{D_2}{\theta}(\frac{1}{\theta}+t_1)\right. \\ &+ \left.\frac{D_2}{\theta}(\frac{e^{\theta(t_1-t_d)}-1}{\theta}+t_d)\right\}+\frac{cD_2}{\theta T}\left\{e^{\theta(t_1-t_d)}-1-\theta(T-t_d)\right\} \\ &+ \frac{cI_c}{T}\left\{\frac{D_1}{2}t_d^2+\frac{D_2}{\theta}t_d(e^{\theta(t_1-t_d)}-1)-D_1(t_dM-\frac{M^2}{2})\right. \\ &- \left.\frac{D_2}{\theta}(\frac{1}{\theta}+t_1)+\frac{D_2}{\theta^2}(e^{\theta(t_1-t_d)}-1+\theta t_d)\right\}-sI_eD_1\frac{M^2}{2T} \\ &+ c_2D_2\delta(\frac{T}{2}-t_1+\frac{t_1^2}{2T}) \end{aligned} \tag{16}$$

Case 2. $M > t_d$

$$TC_2(t_1,T)=\begin{cases} TC_{2,1}(t_1,T) & T\le t_d<M \\ TC_{2,2}(t_1,T) & t_d<T\le M \\ TC_{2,3}(t_1,T) & t_d<M<T \end{cases} \tag{17}$$

where

$$\begin{aligned} TC_{2,1}(t_1,T) &= \frac{1}{T}(k + C_H + P - E + C_s) \\ &= \frac{k}{T} + cD_1 h\frac{T}{2} + sI_e D_1\left\{\frac{T}{2} - M\right\} + c_2 D_2\delta(\frac{T}{2} - t_1 + \frac{t_1^2}{2T}) \end{aligned} \quad (18)$$

$$\begin{aligned} TC_{2,2}(t_1,T) &= \frac{1}{T}(k + cd(T_2) + C_H + P - E + C_s) \\ &= \frac{k}{T} + \frac{ch}{T}\left\{D_1\frac{t_d^2}{2}\frac{D_2 t_d}{\theta}e^{\theta(t_1-t_d)} + \frac{D_2}{\theta^2}e^{\theta(t_1-t_d)} - \frac{D_2}{\theta^2}\right\} \\ &\quad -\frac{ch}{T}\frac{D_2}{\theta}t_1 + \frac{cD_2}{\theta T}(e^{\theta(t_1-t_d)} - 1) - cD_2(1 - \frac{t_d}{T}) \\ &\quad - sI_e\left\{D_1\frac{t_d^2}{2T} + \frac{D_2}{2T}(T^2 - t_d^2) + \frac{D_1}{T}t_d M - D_1 t_d\right\} \\ &\quad - sI_e D_2(M - T - \frac{Mt_d}{T} + t_d) + c_2 D_2\delta(\frac{T}{2} - t_1 + \frac{t_1^2}{2T}) \end{aligned} \quad (19)$$

$$\begin{aligned} TC_{2,3}(t_1,T) &= \frac{1}{T}(k + cd(T_2) + C_H + P - E + C_s) \\ &= \frac{k}{T} + \frac{ch}{T}\left\{\frac{D_1}{2}t_D^2 + \frac{D_2}{\theta}t_d e^{\theta(t_1-t_d)} + \frac{D_2}{\theta^2}e^{\theta(t_1-t_d)}\right\} \\ &\quad - ch\frac{D_2}{T}(\frac{1}{\theta^2} + \frac{t_1}{\theta}) + \frac{cD_2}{\theta T}(e^{\theta(t_1-t_d)} - 1) - cD_2(1 - \frac{t_d}{T}) \\ &\quad + cI_c\frac{D_2}{T}\left\{\frac{1}{\theta^2} - \frac{t_1}{\theta} + \frac{e^{\theta(t_1-M)}}{\theta^2} + \frac{M}{\theta}\right\} + c_2 D_2\delta(\frac{T}{2} - t_1 + \frac{t_1^2}{2T}) \\ &\quad - \frac{sI_e}{2T}\{D_1 t_d^2 + D_2(M^2 - t_d^2)\} \end{aligned} \quad (20)$$

In the next section, our objective is to determine the optimal values of t_1^* and T^* such that $TC_1(t_1,T)$ and $TC_2(t_1,T)$ which attains minimum.

5. Analysis and optimization

(a) The necessary conditions for $TC_{1,1}(t_1,T)$ to be minimized are:

$$\frac{\partial TC_{1,1}(t_1,T)}{\partial t_1} = c_2 D_2\delta(-1 + \frac{t_1}{T}) = 0 \quad (21)$$

$$\frac{\partial TC_{1,1}(t_1,T)}{\partial T} = -\frac{k}{T^2}+cD_1\frac{h}{2}+sI_e\frac{D_1}{2}+c_2D_2\delta(\frac{1}{2}-\frac{t_1^2}{2T^2})=0 \quad (22)$$

Let t_1^* and T^* be the solutions of Eqs.(21) and (22), $H_{1,1}(t_1^*,T^*)$ be the Hessian Matrix of $TC_{1,1}(t_1,T)$ evaluated at t_1^* and T^*. It is known that if this matrix is positive definite, then the solution (t_1^*,T^*) is an optimal solution. From Eqs. (21) and (22), we further have:

$$\left.\frac{\partial^2 TC_{1,1}(t_1,T)}{\partial t_1^2}\right|_{(t_1^*,T^*)} = c_2D_2\frac{\delta}{T}>0 \quad (23)$$

$$\left.\frac{\partial^2 TC_{1,1}(t_1,T)}{\partial T\partial t_1}\right|_{(t_1^*,T^*)} = c_2D_2\delta(\frac{-t_1}{T^2}) \quad (24)$$

$$\left.\frac{\partial^2 TC_{1,1}(t_1,T)}{\partial T^2}\right|_{(t_1^*,T^*)} = \frac{2k}{T^3}+c_2D_2\delta(\frac{t_1^2}{T^3})>0 \quad (25)$$

It was noted from Eqs.(23) – (25) that

$$\left.\left\{\frac{\partial^2 TC_{1,1}(t_1,T)}{\partial t_1^2}\frac{\partial^2 TC_{1,1}(t_1,T)}{\partial T^2}-\left(\frac{\partial^2 TC_{1,1}(t_1,T)}{\partial T\partial t_1}\right)^2\right\}\right|_{(t_1^*,T^*)} = c_2D_2\frac{\delta}{T}(\frac{2k}{T^3}+\frac{c_2D_2\delta}{T}) -(\frac{c_2D_2\delta t_1}{T^2})^2>0 \quad (26)$$

Holds, which implies that the matrix $H_{1,1}(t_1^*, T^*)$ is positive definite and (t_1^*, T^*) is the optimal solution of $TC_{1,1}(t_1,T)$.

(b) Accordingly, the necessary conditions for $TC_{1,2}(t_1,T)$ to be minimized are:

$$\frac{\partial TC_{1,2}(t_1,T)}{\partial t_1} = cI_c\left\{D_2e^{\theta(t_1-t_d)}-\frac{D_2M}{T}e^{\theta(t_1-t_d)}\right\} +c_2D_2\delta(-1+\frac{t_1}{T})=0 \quad (27)$$

$$\frac{\partial TC_{1,2}(t_1,T)}{\partial T} = -\frac{k}{T^2} + cD_1\frac{h}{2} + sI_eD_1\frac{M^2}{2T^2} + c_2D_2\delta(\frac{1}{2} - \frac{t_1^2}{2T^2}) + cI_c\left\{\frac{-D_1}{2} + \frac{D_1}{T^2}(t_dM - \frac{M^2}{2}) + \frac{D_2M}{\theta T^2}(e^{\theta(t_1-t_d)} - 1)\right\} = 0 \quad (28)$$

Let t_1^* and T^* be the solutions of Eqs. (27) And (28), H1, 2(t_1^*, T^*) be the Hessian Matrix of $TC_{1,2}(t_1,T)$ evaluated at t_1^* and T^*. It is known that if this matrix is positive definite, then the solution (t_1^*, T^*) is an optimal solution. From Eqs. (27) And (28), we further have:

$$\left.\frac{\partial^2 TC_{1,2}(t_1,T)}{\partial t_1^2}\right|_{(t_1^*,T^*)} = cI_cD_2\theta e^{\theta(t_1-t_d)}(1 - \frac{M}{T}) + c_2\frac{\delta}{T} > 0 \quad (29)$$

$$\left.\frac{\partial^2 TC_{1,2}(t_1,T)}{\partial T\partial t_1}\right|_{(t_1^*,T^*)} = cI_cD_2\frac{M}{T^2}e^{\theta(t_1-t_d)} - c_2D_2\delta\frac{t_1}{T^2} \quad (30)$$

$$\left.\frac{\partial^2 TC_{1,2}(t_1,T)}{\partial T^2}\right|_{(t_1^*,T^*)} = \frac{cD_1}{T}(h - I_c) + c_2D_2\frac{\delta}{T} > 0 \quad (31)$$

if $h > I_c$.

It was noted from Eqs. (29) – (31) that

$$\left.\left\{\frac{\partial^2 TC_{1,2}(t_1,T)}{\partial t_1^2}\frac{\partial^2 TC_{1,2}(t_1,T)}{\partial T^2} - \left(\frac{\partial^2 TC_{1,2}(t_1,T)}{\partial T\partial t_1}\right)^2\right\}\right|_{(t_1^*,T^*)} = c_2D_1D_2c\frac{\delta}{T^2}(h - I_c) + (c_2D_2\frac{\delta}{T})^2(1 - \frac{t_1^2}{T^2}) + cI_cD_2\theta e^{\theta(t_1-t_d)}(1 - \frac{M}{T})\{\frac{cD_1}{T}(h - I_c) + c_2D_2\frac{\delta}{T}\} + 2cI_cMD_2c_2\delta e^{\theta(t_1-t_d)}\frac{t_1}{T^4} - (cI_cMe^{\theta(t_1-t_d)}\frac{D_2}{T^2})^2 > 0 \quad (32)$$

Holds, which implies that the matrix $H_{1,2}(t_1^*, T^*)$ is positive definite and (t_1^*, T^*) is the optimal solution of $TC_{1,2}(t_1,T)$.

(c) The necessary conditions for $TC_{1,3}(t_1, T)$ to be minimized are:

$$\begin{aligned}\frac{\partial TC_{1,3}(t_1,T)}{\partial t_1} &= \frac{ch}{T}\left\{D_2 t_d e^{\theta(t_1-t_d)} - \frac{D_2}{\theta} + \frac{D_2}{T}e^{\theta(t_1-t_d)}\right\} + \frac{cD_2}{T}e^{\theta(t_1-t_d)} \\ &+\frac{cI_cD_2}{T}\left\{(t_d e^{\theta(t_1-t_d)}) - Me^{\theta(t_1-t_d)} - \frac{1}{\theta}\right\} \\ &+\frac{cI_cD_2}{T\theta}e^{\theta(t_1-t_d)} + c_2D_2\delta(-1+\frac{t_1}{T}) = 0\end{aligned} \tag{33}$$

$$\begin{aligned}\frac{\partial TC_{1,3}(t_1,T)}{\partial T} &= -\frac{k}{T^2} - \frac{ch}{T^2}\left\{\frac{D_1t_d^2}{2} + D_2t_d\frac{e^{\theta(t_1-t_d)}}{\theta} - \frac{D_2}{\theta^2} - \frac{D_2t_1}{\theta}\right\} \\ &-\frac{ch}{T^2}\frac{D_2}{T^2}e^{\theta(t_1-t_d)} - \frac{cD_2}{T^2}\left\{\frac{e^{\theta(t_1-t_d)}}{\theta} - \frac{1}{\theta} + t_d\right\} \\ &-\frac{cI_c}{T^2}\left\{D_1\frac{t_d^2}{2} + D_2t_d\frac{e^{\theta(t_1-t_d)}}{\theta} - D_2\frac{t_d}{\theta} - D_1t_dM\right\} \\ &-\frac{cI_c}{T^2}\left\{D_1\frac{M^2}{2} + D_2M\frac{e^{\theta(t_1-t_d)}}{\theta} + D_2\frac{M}{\theta} - \frac{D_2}{\theta^2} - D_2\frac{t_1}{\theta}\right\} \\ &-\frac{cI_c}{T^2}\left\{D_2\frac{e^{\theta(t_1-t_d)}}{\theta^2} + D_2\frac{t_d}{\theta}\right\} + sI_eD_1\frac{M^2}{2T^2} + c_2D_2\frac{\delta}{2}(1-\frac{t_1^2}{T^2}) = 0\end{aligned} \tag{34}$$

Let t_1^* and T^* be the solutions of Eqs. (33) and (34), $H_{1,3}(t_1^*,T^*)$ be the Hessian Matrix of $TC_{1,3}(t_1,T)$ evaluated at t_1^* and T^*. It is known that if this matrix is positive definite, then the solution (t_1^*,T^*) is an optimal solution. From Eqs. (33) and (34), we further have:

$$\begin{aligned}\left.\frac{\partial^2 TC_{1,3}(t_1,T)}{\partial t_1^2}\right|_{(t_1^*,T^*)} &= \frac{D_2}{T}\{ch + cI_c + \frac{c_2\delta}{\theta}(T-t_1)\} \\ &+c_2D_2\frac{\delta}{T} > 0\end{aligned} \tag{35}$$

$$\left.\frac{\partial^2 TC_{1,3}(t_1,T)}{\partial T \partial t_1}\right|_{(t_1^*,T^*)} = -c_2 D_2 \frac{\delta}{T} \tag{36}$$

$$\left.\frac{\partial^2 TC_{1,3}(t_1,T)}{\partial T^2}\right|_{(t_1^*,T^*)} = c_2 D_2 \frac{\delta}{T} > 0 \tag{37}$$

It was noted from Eqs. (35) – (37) that

$$\left.\left\{\frac{\partial^2 TC_{1,3}(t_1,T)}{\partial t_1^2}\frac{\partial^2 TC_{1,3}(t_1,T)}{\partial T^2} - \left(\frac{\partial^2 TC_{1,3}(t_1,T)}{\partial T \partial t_1}\right)^2\right\}\right|_{(t_1^*,T^*)} = c_2 D_2 \tfrac{\delta}{T}\left\{ch\tfrac{D_2}{T} + cI_c\tfrac{D_2}{T} + c_2 D_2 \theta\delta(1-\tfrac{t_1}{T})\right\} > 0 \tag{38}$$

holds, which implies that the matrix $H_{1,3}(t_1{}^*,T^*)$ is positive definite and $(t_1{}^*,T^*)$ is the optimal solution of $TC_{1,3}(t_1,T)$.

Similarly, our objective is to determine the optimal values of $t_1{}^*$ and T^* such that $TC_2(t_1,T)$ is minimum.

(d) The necessary conditions for $TC_{2,1}(t_1,T)$ to be minimized are:

$$\frac{\partial TC_{2,1}(t_1,T)}{\partial t_1} = c_2 D_2 \delta(-1+\frac{t_1}{T}) = 0 \tag{39}$$

$$\frac{\partial TC_{2,1}(t_1,T)}{\partial T} = -\frac{k}{T^2} + cD_1 h\frac{T}{2} + sI_e\frac{D_1}{2} + \frac{c_2 D_2 \delta}{2}(1-\frac{t_1^2}{T^2}) = 0 \tag{40}$$

(e) The necessary conditions for $TC_{2,2}(t_1,T)$ to be minimized are:

$$\frac{\partial TC_{2,2}(t_1,T)}{\partial t_1} = \frac{chD_2}{T\theta}\left\{(e^{\theta(t_1-t_d)}(t_d\theta+1))-1\right\}+\frac{cD_2}{T}e^{\theta(t_1-t_d)} + c_2D_2\delta(-1+\frac{t_1}{T})=0 \tag{41}$$

$$\frac{\partial TC_{2,2}(t_1,T)}{\partial T} = -\frac{k}{T^2}-\frac{ch}{T^2}\left\{D_1\frac{t_d^2}{2}+D_2e^{\theta(t_1-t_d)}(\frac{t_d}{\theta}+\frac{1}{\theta^2})-D_2(\frac{1}{\theta^2}+\frac{t_1}{\theta})\right\} -\frac{sI_e}{2T^2}\left\{-D_1t_d^2+D_2T^2+d_2t_d^2-2D_1Mt_D-2T^2+2Mt_d\right\} +c_2D_2\frac{\delta}{2}(1-\frac{t_1^2}{T^2})-\frac{cD_2}{\theta T^2}(e^{\theta(t_1-t_d)}-1)-cD_2\frac{t_d}{T^2}=0 \tag{42}$$

f) The necessary conditions for $TC_{2,3}(t_1,T)$ to be minimized are:

$$\frac{\partial TC_{2,3}(t_1,T)}{\partial t_1} = ch\frac{D_2}{T}\left\{t_de^{\theta(t_1-t_d)}+\frac{e^{\theta(t_1-t_d)}}{\theta}-\frac{1}{\theta}\right\}+c_2D_2\delta(-1+\frac{t_1}{T}) +\frac{cD_2}{T}e^{\theta(t_1-t_d)}+\frac{cI_cD_2}{\theta T}\{-1+e^{\theta(t_1-M)}\}=0 \tag{43}$$

$$-\frac{cD_2}{\theta T^2}(e^{\theta(t_1-t_d)}-1)-cD_2\frac{t_d}{T^2}+c_2D_2\delta(\frac{1}{2}-\frac{t_1^2}{2T^2}) -\frac{cI_cD_2}{T^2}\left\{\frac{-1}{\theta^2}-\frac{t_1}{\theta}+\frac{e^{\theta(t_1-M)}}{\theta^2}+\frac{M}{\theta}\right\} -\frac{sI_e}{2T^2}(D_1t_d^2+D_2(M^2-t_d^2))=0 \tag{44}$$

By using the similar argument as in the proof of (a),(b) and (c) we can show that $TC_2(t_1,T)$ attains minimum.

It can be obtained numerical solution of t_1^* and T^* by solving Eqs. (21)–(22), Eqs. (27)–(28), Eqs. (33)–(34), Eqs. (39)–(40), Eqs. (41)–(42), Eqs. (43)–(44) respectively. In the next section, we provide an optimization algorithm to find the optimal solutions.

5.1 Algorithm

Step 1.

Compare the values of M and t_d. If $M \le t_d$, then go to **Step 2.** Otherwise, if $M > t_d$ go to **Step 6.**

Step 2.

Solving Eqs. (21)–(22) by Newton-Raphson Method, getting the optimal solution t_1^* and T^*.

If $T < M \le t_d$, then $TC^* = TC_{1,1}(t_1^*,T^*)$. Otherwise go to **Step 3.**

Step 3.

Solving Eqs. (27)–(28) by Newton-Raphson Method, getting the optimal solution t_1^* and T^*.

If $M < T \le t_d$, then $TC^* = TC_{1,2}(t_1^*,T^*)$. Otherwise go to **Step 4.**

Step 4.

Solving Eqs. (33)–(34) by Newton-Raphson Method, getting the optimal solution t_1^* and T^*.

If $M \le t_d < T$, then $TC^* = TC_{1,3}(t_1^*,T^*)$. Otherwise go to **Step 5.**

Step 5.

Let $(t_1^*,T^*) = \min\{TC_{1,1}(t_1^*,T^*),TC_{1,2}(t_1^*,T^*),TC_{1,3}(t_1^*,T^*)\}$.

Step 6.

Solving Eqs. (39) – (40) by Newton-Raphson Method, getting the optimal solution t_1^* and T^*. If $T \le t_d < M$, then $TC^* = TC_{2,1}(t_1^*,T^*)$. Otherwise go to **Step 7.**

Step 7.

Solving Eqs. (41) – (42) by Newton-Raphson Method, getting the optimal solution t_1^* and T^*. If $t_d < T \le M$, then $TC^* = TC_{2,2}(t_1^*,T^*)$. Otherwise go to **Step 8.**

Step 8.

Solving Eqs. (43) – (44) by Newton-Raphson Method, getting the optimal solution t_1^* and T^*. If $t_d < M < T$, then $TC^* = TC_{2,3}(t_1^*,T^*)$. Otherwise go to **Step 9.**

Step 9.

Let $(t_1^*,T^*) = \min\{TC_{2,1}(t_1^*,T^*),TC_{2,2}(t_1^*,T^*),TC_{2,3}(t_1^*,T^*)\}$.

6. Numerical examples

Example 1: k = 100, s = 45, c = 40, D_2 = 2000, D_2 = 500, h = 0.20, M = 0.0658, t_d = 0.0767, I_C = 0.12, I_e = 0.30, θ = 0.40, c_2 = 30, δ = 15, with appropriate units.

According to the algorithm in previous section, the optimal values are obtained as follows:

TC^*=1157.3944, Q = 149.29872, t_1^*= 0.0681107, T^*= 0.0681355.

Example 2: k = 60, s = 50, c = 45, D_1 = 1500, D_2 = 400, h = 0.20, M = 0.0384, t_d = 0.0767, I_c = 0.13, I_e = 0.11, θ = 0.10, c_2 = 30, δ = 15, with appropriate units.

According toalgorithm in previous section, the optimal values are obtained as follows:

TC^* = 5587.8719, Q = 1710.9402, t_1^* = 1.6368412, T^* = 1.7902611.

Example 3: k = 450, s = 50, c = 40, D_1 = 1000, D_2 = 400, h = 0.14, M = 0.0384, t_d =0.0767, I_c = 0.04, I_e = 0.11, θ = 0.50, c_2 = 30, δ = 15, with appropriate units.

According toalgorithm in previous section, the optimal values are obtained as follows:

TC^* =1104.897, Q = 862.44821, $t1^*$ = 0.4356446, T^* =0.5403912.

Example 4: k = 200, s = 60, c = 50, D_1 = 2000, D_2 = 1000, h = 0.12, M = 0.0959, t_d = 0.0384, I_c = 0.04, I_e

= 0.11, θ = 0.20, c_2 = 30, δ = 15, with appropriate units.

According toalgorithm in previous section, the optimal values are obtained as follows:

TC^* = 10320.937, Q = 3932.8774, t_1^* =1.3851085, $T*$ =1.5391462.

Example 5: k = 140, s = 60, c = 50, D_1 = 2000, D_2 = 1500, h = 0.10, M = 0.0767, t_d = 0.0384, I_c = 0.13, I_e = 0.11, θ = 0.10, c_2 = 30, δ = 15, with appropriate units.

According to the algorithm in previous section, the optimal values are obtained as follows:

TC^* =7195.0516, Q = 3356.0652, $t1^*$ = 0.9005981, T^* = 1.0485361.

7 Sensitivity analyses

In this section, we study the effects of changes in the parameters δ and c_2 on the optimal policies of Example 3. We obtain the computational results as shown in the following Table 1 and Table 2 respectively.

Table 1. Effects of changes in parameter of the model

δ	t_1^*	T^*	TC^*	Q
15	0.2847474	0.5403912	1104,897	862.44821
25	0.0843507	0.138619	280.47173	622.44914
30	0.1278126	0.1743811	764.67392	656.23053
35	0.1548851	0.1955318	1071.6459	677.64718
40	0.1737777	0.209795	1288.3425	692.76572
45	0.1878229	0.2201392	1450.7723	704.10058
55	0.2074255	0.2342125	1697.3862	720.05097
70	0.2255587	0.2468598	1892.8683	734.94619

Table 2. Effects of changes in parameter of the model

$c2$	t_1^*	T^*	TC^*	Q
30	0.4356446	0.5403912	1104.897	862.44821
35	0.3147846	0.3995596	581.67228	686.48405
45	0.0487349	0.1075921	1340.3273	418.735
60	0.1278126	0.1743811	764.67392	376.81953
70	0.1548851	0.1955318	1071.6457	352.47358

We study the effects of changes in the parameters k,D_1,D_2,M,θ, and t_d on the optimal policies of Example 3. The setting values of k,D_1,D_2,M,θ, and t_d are assumed as $k \in \{450,500,550,650\}$, $D_1 \in \{1000,1500,2000,2500\}$, $D_2 \in \{400,800,1200,1600\}$, $M \in \{14,19,24,27\}$, $\theta \in \{0.5,1,1.5,2\}$ and $t_d \in \{28,33,38,43\}$. According to the algorithm in previous section, we obtain the computational results as shown in the following Table 3.

Table 3. Effects of changes in parameter of the model

Parameters	Change in parameters	t_1^*	T^*	TC^*	Q
k	450	0.4356446	0.5403912	1104.897	862.44821
	500	0.3684702	0.4698451	776.11151	810.59999
	550	0.265596	0.3619533	267.10987	734.10987
	650	0.0232814	0.1100461	-382.35585	576.20359
D_1	1000	0.4356446	0.5403912	1104.897	862.44821
	1500	0.427899	0.5322537	1067.5505	894.74667
	2000	0.4191832	0.5230959	1024.29815	926.29815
	2500	0.4105978	0.5140779	982.47442	957.9857
D_2	400	0.4356446	0.5403912	1104.897	862.44821
	800	0.6192524	0.7336245	3974.8011	1947.7849
	1200	0.6606639	0.7772913	6543.7251	2989.7824
	1600	0.679776	0.7974564	9083.4742	4027.2264
M (in days)	0	0.4308611	0.535516	1106.3901	859.6112
	14	0.4356446	0.5403912	1104.897	862.44821
	19	0.4390626	0.5439295	1112.9498	864.80738
	24	0.442942	0.5479511	1122.6629	867.52237
	27	0.4456347	0.5507477	1130.2605	869.44017
θ	0.15	1.0384271	1.1494353	2229.9469	1156.5714
	0.25	0.7528071	0.8617322	1854.8936	1024.8936
	0.35	0.588192	0.6954069	1540.4285	944.0435
	0.5	0.4356446	0.5403912	1104.897	862.44821
	0.75	0.2824937	0.3825368	364.17708	765.96858
t_d (in days)	28	0.4356446	0.5403912	1104.897	862.44821
	33	0.4095444	0.5122952	865.70383	845.3178
	38	0.3768394	0.4773002	596.67321	823.75442
	43	0.332927	0.4305929	274.42004	794.6492

Based on the computational results are shown in Table 3, we obtain the following managerial insights :

1. It can be found that each of t_1^*,T^*,TC^* and Q decreases with an increase of ordering cost, for fixed other parameters. It implies that, for higher the ordering cost, the retailer will order only lower quantity of items.
2. When the demand before the deterioration sets in increases and other parameters remain unchanged then each of t_1^*,T^* and TC^* decreases while the ordering quantity increases. This implies that the retailer would order more quantity which is realistic in real-time situation.
3. All the optimal values t_1^*,T^*,TC^* and Q increases, for other parameters remain unchanged when the demand after deterioration sets in, which is reasonable from economical point of

view.

4. It is observed that each of t_1^*, T^* and Q increases while TC^* increases when all other parameters remain unchanged which implies that the longer the trade credit period supplier offer is, longer the replenishment cycle, higher the ordering quantity and the total cost will be.
5. When θ increases, for fixed other parameters, all optimal values t_1^*, T^*, TC^* and Q decreases. It implies that the rate deterioration increases, the retailer would order only lower quantity and hence the total cost will be lowered.
6. When the length of time the product has not deterioration increases, and the other parameters remain unchanged, the ordering quantity decreases. Also, if the retailer can extend the length of time the product has no deterioration over the time period, the total costs will be reduced.

8. Conclusion

In this study, we develop an inventory model for deteriorating items with shortages which are completely backlogged when the supplier offers permissible delay in payments. This will be helpful to the retailer to find the optimal replenishment policy under various situations. We also provide an algorithm to characterize the optimal solutions and optimal replenishment cycle time. It can be found that the ordering cost influences when determining an optimal replenishment policies from the theoretical results. Further, the higher the ordering cost is, the lower the replenishment cycle time and lower the ordering quantity. Also, we provide several numerical examples to illustrate the theoretical results in which the present model deals two cases where the demand before and after the deterioration sets in.

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