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Analysis of an M/G/1 Queue with Multiple Vacations, N-policy, Unreliable Service Station and Repair Facility Failures

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Abstract
This paper studies an M/G/1 repairable queueing system with multiple vacations and N-policy, in which the service station is subject to occasional random breakdowns. When the service station breaks down, it is repaired by a repair facility. Moreover, the repair facility may fail during the repair period of the service station. The failed repair facility resumes repair after completion of its replacement. Under these assumptions, applying a simple method, the probability that the service station is broken, the rate of occurrence of breakdowns of the service station, the probability that the repair facility is being replaced and the rate of occurrence of failures of the repair facility along with other performance measures are obtained. Following the construction of the long-run expected cost function per unit time, the direct search method is implemented for determining the optimum threshold N* that minimises the cost function.

Keywords: M/G/1 repairable queue; multiple vacations; min (N, V)-policy; reliability measure; cost function.

1. Introduction

In this paper, we consider an M/G/1 repairable queueing system with an unreliable service station, an unreliable repair facility operating under multiple vacations and N-policy simultaneously. Under this type of control policy, the server leaves for a vacation when the system becomes empty. Upon returning from his/her vacation, if there are one or more customers queueing up for service in the system, the server starts providing service immediately. Otherwise, if there are no customers waiting for service in the system, the server leaves for another vacation immediately.

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This pattern continues until the system is not empty. Whenever the number of customers waiting for service in the system has accumulated to a predefined threshold value N, the server interrupts his/her vacation and returns to the system to serve customers immediately. This commences a new busy period and the server works until the system becomes empty. We call this discipline multiple vacations with min (N, V)-policy. We are interested in when the server should start his/her service due to the number of customers waiting in the system during his/her vacation period, that is, how to determine the threshold N.

Vacation queueing models with control policy have received a lot of attention due to their abundant applications, especially in the production/inventory systems and the manufacturing systems. For example, Kella (1989), Lee et al. (1994), Gakis et al. (1995), Alfa and Frigui (1996), Reddy (1998) and Baek et al. (2014). Alfa and Li (2000) investigated the optimum (N,T)-policy for an M/G/1 queue with cost structure where the (N,T)-policy means that the system reactivates as soon as N customers are present or the waiting time of the leading customer reaches a predefined time T. Later, Hur et al. (2003) studied an M/G/1 system with N-policy and T-policy. In their work, the server takes a vacation of fixed length T when the system becomes empty. After T time units, if the server finds customers waiting in the system, he/she starts the service. Otherwise, he/she leaves for another vacation of length T. This continues until the system is not empty. Once the number of customers waiting for service reaches N, the server interrupts his/her vacation and starts providing service immediately. Although the fixed length T of vacation time greatly simplifies analysis, it makes it difficult to approximate the real world scenarios precisely. Thus, here a random amount V of vacation time is assumed.

Accomplishing the development of manufacturing and communication systems, the reliability analysis for the queueing system with an unreliable service station has been done by a considerable amount of work in the past, and successfully used in various applied problems. Among some excellent papers in this area are those by Cao and Cheng (1982), Tang (1997), Wang et al. (2008), Choudhury and Deka (2008) and Gao and Wang (2014). We observe that authors usually supposed the service station breaks down during service period of customers, but the repair facility for the broken service station does not fail. However, in practice, the repair facility may fail during the repair period of the service station due to temperature changes, voltage fluctuations and human operational errors. Thus, repairable queueing systems with replaceable repair facility are more general. Once the repair facility fails, it should be replaced by another new one. In reliability theory, models with replaceable repair facility have been extensively studied, such as by Zhang and Wu (2009), Tang (2010), Yu et al. (2013) and others. So, the case of replaceable repair facility is taken into account in this paper. It is worth noticing that this work definitely differs from the previously studied models, since reliability aspects are applied not only to the service station, but also to the repair facility.

This paper is arranged as follows. Section 2 gives the model description and some preliminaries. Reliability measures of the service station and the repair facility are derived in Sections 3 and Sections 4, respectively. Other performance measures are calculated in Section 5. In Section 6, a long-run expected cost function per unit time is constructed. Section 7 gives conclusions.

2. Model description and some preliminaries

2.1. Basic assumptions

The detailed assumptions of the system are described as follows.

Assumption 1. Initially, a queueing system with one server, a new service station and a new repair facility is installed, and there exists $i\left(i \ge 0\right)$ customers in the system. At time t=0, the server does not take vacation if the system is empty.

Assumption 2. Customers arrive to the system at the instants $\varsigma_n \left(n = 1, 2, \cdots\right)$. The inter-arrival time $\tau_n = \varsigma_{n+1} - \varsigma_n \left(n = 0, 1, \cdots; \tau_0 = 0\right)$ are mutually independent and identically distributed random variables with exponential distribution $F(t) = 1 - \exp(-\lambda t)$, $\lambda > 0$, $t \ge 0$. The service times $\{\chi_n, n \ge 1\}$ of the customers provided by the server are independent identically random variables following general distribution $G(t)$, $t \ge 0$ with mean service time $0 < 1/\mu = \int_0^\infty t dG(t) < \infty$.

Assumption 3. The operating time X of the service station is governed by an exponential distribution $X(t) = 1 - \exp(-\alpha t)$, $\alpha \ge 0$, $t \ge 0$. When the service station breaks down, it is repaired by a repair facility, while the service to the customer being served will be stopped. The service station restarts its service to the customer as soon as the repair is finished. Furthermore, the repaired service station is as good as new. Assume that the repair time Y of the service station follows a general distribution $Y(t)$, $t \ge 0$, with mean repair time $0 \le 1/\beta = \int_0^\infty t dY(t) < \infty$.

Assumption 4. The repair facility may fail during the repair time Y. If the repair facility fails, it should be replaced by another new one, while the broken service station has to wait. The repair facility resumes repair after completion of its replacement. Moreover, the working time R of the repair facility follows an exponential distribution $R(t) = 1 - \exp(-rt)$, $r \ge 0$, $t \ge 0$, while the replacement time W follows an arbitrary distribution $W(t)$, $t \ge 0$, with mean replacement time $0 \le 1/w = \int_0^\infty t dW(t) < \infty$.

Assumption 5. The server leaves for a vacation when the system becomes empty. Upon returning from his/her vacation, if there exists customers in the system, he/she starts the service. Otherwise, he leaves for another vacation. Whenever the number of customers waiting in the system reaches N, the server interrupts his/her vacation and starts providing service immediately. The vacation time V follows an arbitrary distribution $V(t), t \ge 0$, with mean vacation time $0 \le 1/v = \int_0^\infty t dV(t) < \infty$.

Assumption 6. The random variables involved in the system are assumed to be independent of each other.

We define the following notations for further use in the sequel.

$\hat{a}(s)$	the Laplace-Stieltjes transform of an arbitrary function $A(t)$, i.e., $\hat{a}(s)=\int_0^\infty e^{-st}dA(t)$
$A^*(s)$	the Laplace transform of an arbitrary function $A(t)$, i.e., $A^*(s)=\int_0^\infty e^{-st}A(t)dt$
$A^{(k)}(t)$	the k -fold convolution of an arbitrary function $A(t)$ with itself, $k \ge 1$, and $A^{(0)}(t)=1,\ t\ge 0$
$\bar{A}(t)$	the complement of the function $A(t)$, i.e., $\bar{A}(t)=1-A(t)$
$A(t)*U(t)$	the convolution product of functions $A(t)$ and $U(t)$, i.e., $A(t)*U(t)=\int_0^t A(t-x)dU(x)=\int_0^t U(t-x)dA(x)$
$\Re e(s)$	the real part of the complex variable s
$N(t)$	the number of customers in the system at time t

2.2. Preliminaries

Denote by $\tilde{Y}_n$ $(n=1,2,\cdots)$ the "generalized repair time" of the service station after the n th broken, where $\tilde{Y}_n$ contains some replacement times of the repair facility owing to its failures. Further, setting $\tilde{Y}_n(t)=P\{\tilde{Y}_n\le t\}, t\ge 0$, and using the method provided in Cao and Cheng (1982), we obtain that

$$\tilde{Y}(t)=\tilde{Y}_n(t)=\sum_{j=0}^{\infty}\int_0^t W^{(j)}(t-x)\frac{(rx)^j}{j!}e^{-rx}dY(x), t\ge 0, \quad (1)$$

which is independent of n. The Laplace-Stieltjes transform of $\tilde{Y}(t)$ is

$$\hat{\tilde{y}}(s)=\sum_{j=0}^{\infty}[\hat{w}(s)]^j\int_0^\infty e^{-(s+r)t}\frac{(rt)^j}{j!}dY(t)=\hat{y}\left(s+r-r\hat{w}(s)\right), \Re e(s)>0, \quad (2)$$

and its expected value is given by

$$E[\tilde{Y}]=-\frac{d}{ds}\left[\hat{\tilde{y}}(s)\right]\Big|_{s=0}=\frac{w+r}{w\beta}. \quad (3)$$

Let $\tilde{\chi}_n$ represent the "generalized service time" of the nth customer, that is, the length of time since the nth customer starts to be served until the service is finished, where $\tilde{\chi}_n$ includes some repair times of the service station owing to its breakdowns during the service period of the customer, and some possible replacement times of the repair facility due to its failures during the repair period of the service station. Moreover, let $\tilde{G}_n(t) = P\{\tilde{\chi}_n \le t\}, t \ge 0$, similar to Equation (1), we have that

$$\tilde{G}(t) = \tilde{G}_n(t) = \sum_{l=0}^{\infty} \int_0^t \tilde{Y}^{(l)}(t-x) \frac{(\alpha x)^l}{l!} e^{-\alpha x} dG(x), t \ge 0, \tag{4}$$

Which is independent of n. Its Laplace-Stieltjes transform is given by

$$\hat{\tilde{g}}(s) = \sum_{l=0}^{\infty} \left[\hat{\tilde{y}}(s)\right]^l \int_0^{\infty} e^{-(s+\alpha)t} \frac{(\alpha t)^l}{l!} dG(t) = \hat{g}\left(s+\alpha-\alpha\hat{\tilde{y}}(s)\right), \Re e(s) > 0, \tag{5}$$

And the mean of the generalized service time is

$$E[\tilde{\chi}] = -\frac{d}{ds}\left[\hat{\tilde{g}}(s)\right]\Big|_{s=0} = \frac{w\beta + \alpha\left(w+r\right)}{\mu w \beta} \quad . \tag{6}$$

The server's "generalized busy period" is the time interval from the server starts service until the system becomes empty. Let $\tilde{b}$ be the length of server's generalized busy period beginning with one customer, and $\tilde{B}(t) = P\{\tilde{b} \le t\}, t \ge 0$.We have the following lemma from Tang (1997)

Lemma 1. For $\Re e(s) > 0$, the $\hat{\tilde{b}}(s)$ is the root with the smallest absolute value of the equation $z = \hat{\tilde{g}}(s+\lambda-\lambda z), |z| < 1$, and

$$\lim_{t\to\infty} \tilde{B}(t) = \lim_{s\to 0^+} \hat{\tilde{b}}(s) = \begin{cases} 1, & \tilde{\rho} \le 1 \\ \sigma, & \tilde{\rho} > 1 \end{cases}, \quad E[\tilde{b}] = \begin{cases} \dfrac{\tilde{\rho}}{\lambda(1-\tilde{\rho})}, & \tilde{\rho} < 1 \\ \infty, & \tilde{\rho} \ge 1 \end{cases},$$

Where $\tilde{\rho} = \dfrac{\lambda\left[\beta w + \alpha\left(w+r\right)\right]}{\mu\beta w}$, σ $(0 < \sigma < 1)$ is the root of the equation $z = \hat{\tilde{g}}(\lambda - \lambda z)$.

Denote by $\tilde{b}^{<i>}$ the server's generalized busy period initiated with $i (i \ge 1)$ customers. Based on the Poisson arrival process, $\tilde{b}^{<i>}$ can be expressed as $\tilde{b}^{<i>} = \tilde{b}_1 + \tilde{b}_2 + \cdots + \tilde{b}_i, i \ge 1$, where $\tilde{b}_1, \tilde{b}_2, \cdots, \tilde{b}_i$ are independent of each other with the same distribution $\tilde{B}(t)$. Moreover, we have that

$$\tilde{B}^{<i>}(t) = P\{\tilde{b}^{<i>} \le t\} = \tilde{B}^{(i)}(t), \quad t \ge 0.$$

We define $\hat{\tau}_i$ to be the i th idle period of the system, that is, the length of time since the system becomes empty until the first customer arrives. Thus, the Poisson arrival process implies $P\{\hat{\tau}_i \le t\} = F(t) = 1 - e^{-\lambda t}, i \ge 1, t \ge 0$.

3. Reliability measures of the service station

3.1. The probability that the service station is broken at time t

First, we consider a classical simple repairable system (see Cao and Cheng, 2006). The operating time X of the system follows an exponential distribution with parameter $\alpha\left(\alpha \ge 0\right)$, while the repair time $\tilde{Y}$ obeys a general distribution (1) above. The system is "as good as new" after being repaired, and restarts to operate immediately. Let

$$I(t) = P\{\text{the system is broken at time } t\}, \quad t \ge 0,$$

$$K(t) = E\{\text{the number breakdowns of the system during } (0,t]\}, t \ge 0.$$

Lemma 2. If $\Re e(s) > 0$, then

$$I^*(s) = \frac{\alpha\left[1 - \hat{\tilde{y}}(s)\right]}{s\left[s + \alpha - \alpha\hat{\tilde{y}}(s)\right]}, \quad \hat{k}(s) = \frac{\alpha}{s + \alpha - \alpha\hat{\tilde{y}}(s)}. \tag{7}$$

We now discuss the reliability measures of the service station. Let

$$\Phi_i(t) = P\{\text{the service station is broken at time } t \left| N(0) = i \right.\}, i = 0,1,\cdots.$$

Theorem 1. The Laplace transform of $\Phi_i(t)\left(i = 0,1,\cdots\right)$ are given by

$$\Phi_0^*(s) = \frac{\alpha\left[1 - \hat{\tilde{y}}(s)\right]\hat{f}(s)}{s\left[s + \alpha - \alpha\hat{\tilde{y}}(s)\right]}\left[1 - \frac{\hat{\tilde{b}}(s)\Pi(s)}{1 - \hat{v}(s+\lambda) - \Delta(s)}\right], \tag{8}$$

$$\Phi_i^*(s) = \frac{\alpha\left[1 - \hat{\tilde{y}}(s)\right]}{s\left[s + \alpha - \alpha\hat{\tilde{y}}(s)\right]}\left[1 - \frac{\hat{\tilde{b}}^i(s)\Pi(s)}{1 - \hat{v}(s+\lambda) - \Delta(s)}\right], \quad i \ge 1, \tag{9}$$

And the steady-state probability that the service station is broken is

$$\Phi = \lim_{t\to\infty}\Phi_i(t) = \lim_{s\to 0^+} s\Phi_i^*(s) = \begin{cases} \dfrac{\lambda\alpha\left(w + r\right)}{\mu\beta w}, & \tilde{\rho} < 1, \\ \dfrac{\alpha\left(w + r\right)}{\alpha\left(w + r\right) + w\beta}, & \tilde{\rho} \ge 1, \end{cases} \tag{10}$$

Which is independent of the initial state $N(0) = i\left(i = 0,1,\cdots\right)$, and $\hat{v}(s+\lambda) = \int_0^\infty e^{-(s+\lambda)t} dV(t)$,

$$\Pi(s)=1-\hat{v}(s+\lambda)-\int_0^\infty e^{-st}\overline{V}(t)dF^{(N)}(t)-\sum_{m=1}^{N-1}\int_0^\infty e^{-(s+\lambda)t}\frac{(\lambda t)^m}{m!}dV(t),$$

$$\Delta(s)=\hat{\tilde{b}}^N(s)\int_0^\infty e^{-st}\overline{V}(t)dF^{(N)}(t)-\sum_{m=1}^{N-1}\int_0^\infty e^{-(s+\lambda)t}\frac{\left(\lambda\hat{\tilde{b}}(s)t\right)^m}{m!}dV(t).$$

Proof. 1) Under the present assumptions, the service station breaks down only in the server's generalized busy period, and it operates at the beginning of every server's generalized busy period. That is, the service station is broken at time t if and only if the time t is in one server's generalized busy period and the service station is broken at time t. Let $s_j=\sum_{j=1}^{k}V_j$, $l_j=\sum_{j=1}^{k}\tau_j, k\geq 1,\ \ s_0=l_0=0$.

For $i=0$

$$\Phi_0(t)=P\{\text{the service station is broken at time } t, \hat{\tau}_1\leq t<\hat{\tau}_1+\tilde{b}_1\}$$

$$+\sum_{k=1}^{\infty}\sum_{m=1}^{N-1}P\{\text{the service station is broken at time } t, \hat{\tau}_1+\tilde{b}_1+s_k\leq t, s_{k-1}<\hat{\tau}_2, s_{k-1}<\hat{\tau}_2+l_{m-1}\leq s_k<\hat{\tau}_2+l_m\}$$

$$+\sum_{k=1}^{\infty}P\{\text{the service station is broken at time } t,$$

$$\hat{\tau}_1+\tilde{b}_1+\hat{\tau}_2+l_{N-1}\leq t, s_{k-1}<\hat{\tau}_2, s_{k-1}<\hat{\tau}_2+l_{N-1}\leq s_k\}$$

$$=\int_0^t P\{\text{the service station is broken at time } t-x, \tilde{b}_1>t-x\geq 0\}dF(x)$$

$$+\sum_{k=1}^{\infty}\sum_{m=1}^{N-1}\int_0^t\int_0^{t-x}\int_0^{t-x-y}\Phi_m(t-x-y)\frac{(\lambda z)^m}{m!}e^{-\lambda(y+z)}dV(z)dV^{(k-1)}(y)d\left[F(x)*\tilde{B}(x)\right]$$

$$+\sum_{k=1}^{\infty}\int_0^t\int_0^{t-x}\int_0^{t-x-y}\Phi_N(t-x-y)\overline{V}(z)e^{-\lambda y}dF^{(N)}(z)dV^{(k-1)}(y)d\left[F(x)*\tilde{B}(x)\right]. \tag{11}$$

Similarly, for $i\geq 1$, we have

$$\Phi_i(t)=P\{\text{the service station is broken at time } t, \tilde{b}^{<i>}>t\geq 0\}$$

$$+\sum_{k=1}^{\infty}\sum_{m=1}^{N-1}\int_0^t\int_0^{t-x}\int_0^{t-x-y}\Phi_m(t-x-y)\frac{(\lambda z)^m}{m!}e^{-\lambda(y+z)}dV(z)dV^{(k-1)}(y)d\tilde{B}^{(i)}(x)$$

$$+\sum_{k=1}^{\infty}\int_0^t\int_0^{t-x}\int_0^{t-x-y}\Phi_N(t-x-y)\overline{V}(z)e^{-\lambda y}dF^{(N)}(z)dV^{(k-1)}(y)d\tilde{B}^{(i)}(x). \tag{12}$$

2) Let $Q_i(t)=P\{\text{the service station is broken at time } t, \tilde{b}^{<i>}>t\geq 0\}$, $i\geq 1$.We decompose

$I(t)$ by server's generalized busy period $\tilde{b}^{<i>}$ (see Fig. 1), then

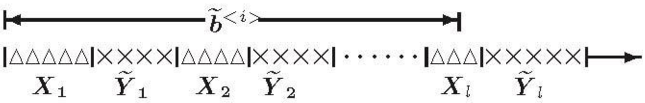

Figure 1. Case of the service station in server's generalized busy period.

$$Q_i(t) = I(t) - \int_0^t I(t-x)d\tilde{B}^{(i)}(x)\,, i \geq 1\,. \tag{13}$$

3) Substituting expression (13) into expressions (11) and (12), and by taking the Laplace transform, it yields that

$$\Phi_0^*(s) = I^*(s)\hat{f}(s)\left[1-\hat{\tilde{b}}(s)\right] + \sum_{m=1}^{N-1}\Phi_m^*(s)\int_0^\infty e^{-(s+\lambda)t}\frac{(\lambda t)^m}{m!}dV(t)\frac{\hat{f}(s)\hat{\tilde{b}}(s)}{1-\hat{v}(s+\lambda)}$$

$$+\Phi_N^*(s)\int_0^\infty e^{-st}\overline{V}(t)dF^{(N)}(t)\frac{\hat{f}(s)\hat{\tilde{b}}(s)}{1-\hat{v}(s+\lambda)}, \tag{14}$$

$$\Phi_i^*(s) = I^*(s)\left[1-\hat{\tilde{b}}^i(s)\right] + \sum_{m=1}^{N-1}\Phi_m^*(s)\int_0^\infty e^{-(s+\lambda)t}\frac{(\lambda t)^m}{m!}dV(t)\frac{\hat{\tilde{b}}^i(s)}{1-\hat{v}(s+\lambda)}$$

$$+\Phi_N^*(s)\int_0^\infty e^{-st}\overline{V}(t)dF^{(N)}(t)\frac{\hat{\tilde{b}}^i(s)}{1-\hat{v}(s+\lambda)}. \tag{15}$$

Furthermore, we obtain expressions (8) and (9) from expressions (14) and (15). The expression (10) is derived from the L'Hospital rule.

3.2. The expected number of the service station breakdowns during (0, t]

For $t \geq 0$, let

$$M_i(t) = E\{\text{the number of service station breakdowns during } (0,t] \,|\, N(0)=i\}, i = 0,1,\cdots.$$

Theorem 2. The Laplace-Stieltjes transform of $M_i(t)\left(i=0,1,\cdots\right)$ are

$$\hat{m}_0(s) = \frac{\alpha\hat{f}(s)}{s+\alpha-\alpha\hat{\tilde{y}}(s)}\left[1-\frac{\hat{\tilde{b}}(s)\Pi(s)}{1-\hat{v}(s+\lambda)-\Delta(s)}\right], \tag{16}$$

$$\hat{m}_i(s) = \frac{\alpha}{s+\alpha-\alpha\hat{\tilde{y}}(s)}\left[1-\frac{\hat{\tilde{b}}^i(s)\Pi(s)}{1-\hat{v}(s+\lambda)-\Delta(s)}\right], i \geq 1\,, \tag{17}$$

And the rate of occurrence of breakdowns of the service station is given by

$$M = \lim_{t\to\infty}\frac{M_i(t)}{t} = \lim_{s\to 0^+} s\widehat{m}_i(s) = \begin{cases} \dfrac{\lambda\alpha}{\mu}, & \tilde{\rho} < 1 \\ \dfrac{\alpha w\beta}{\alpha(w+r)+w\beta}, & \tilde{\rho} \ge 1 \end{cases}, \tag{18}$$

Which is independent of the initial state $N(0)=i(i\ge 0)$, $\Pi(s)$ and $\Delta(s)$ are given in Theorem 1.

Proof. Similar to the proof of Theorem 5 in Tang (1997).

4. Reliability measures of the repair facility

4.1. The probability that the repair facility is being replaced

The "generalized busy period" of the repair facility denotes the time interval from the instant when it starts to repair the broken service station to the moment that the repair is completed, where it includes some possible replacement times of the repair facility due to its failures. Obviously, the "generalized busy period" of the repair facility is the "generalized repair time" of the service station.

We first consider a classical one-unit system (see Cao and Cheng, 2006). The working time R of the unit follows an exponential distribution with parameter $r(r\ge 0)$. When the unit fails, it should be replaced by a new one. Let W be the replacement time of the unit, which is governed by an arbitrary distribution $W(t)$, $t\ge 0$. Let

$$C(t) = P\{\text{the unit is failed at time } t\},\ t\ge 0,$$

$$D(t) = E\{\text{the number of the unit failures during } (0,t]\},\ t\ge 0.$$

Lemma 3. If $\Re e(s) > 0$, then

$$C^*(s) = \frac{r[1-\widehat{w}(s)]}{s[s+r-r\widehat{w}(s)]}, \qquad \widehat{d}(s) = \frac{r}{s+r-r\widehat{w}(s)}. \tag{19}$$

We now turn our interest to the reliability measures of the repair facility. First, set

$$\Gamma_i(t) = P\{\text{the repair facility is being replaced at time } t \mid N(0)=i\},\ i=0,1,\cdots.$$

Theorem 3. The Laplace transform of $\Gamma_i(t)(i=0,1,\cdots)$ are given by

$$\Gamma_0^*(s) = \frac{r[1-\widehat{w}(s)]}{s[s+r-r\widehat{w}(s)]}\frac{\alpha\left[1-\widehat{\widehat{y}}(s)\right]\widehat{f}(s)}{s+\alpha-\alpha\widehat{\widehat{y}}(s)}\left[1-\frac{\widehat{\widehat{b}}(s)\Pi(s)}{1-\widehat{v}(s+\lambda)-\Delta(s)}\right], \tag{20}$$

$$\Gamma_i^*(s) = \frac{r[1-\widehat{w}(s)]}{s[s+r-r\widehat{w}(s)]}\frac{\alpha\left[1-\widehat{\widehat{y}}(s)\right]}{s+\alpha-\alpha\widehat{\widehat{y}}(s)}\left[1-\frac{\widehat{\widehat{b}}^{i}(s)\Pi(s)}{1-\widehat{v}(s+\lambda)-\Delta(s)}\right], i\ge 1, \tag{21}$$

And the steady-state probability that the repair facility is being replaced is

$$\Gamma = \lim_{t\to\infty}\Gamma_i(t) = \lim_{s\to 0^+} s\Gamma_i^*(s) = \begin{cases} \dfrac{\lambda\alpha r}{\mu\beta w}, & \tilde{\rho}<1 \\ \dfrac{\alpha r}{\alpha\left(w+r\right)+w\beta}, & \tilde{\rho}\geq 1 \end{cases}, \tag{22}$$

Which is independent of the initial state $N(0)=i\left(i\geq 0\right)$, $\Pi(s)$ and $\Delta(s)$ are given in Theorem 1.

Proof. 1) Under the given assumptions, the repair facility is perfect when it does not repair broken service station. That is, it is being replaced at time t if and only if the time t is within one server's generalized busy period and within one generalized repair time of the service station. For $i=0$, we get

$$\Gamma_0(t) = \int_0^t \Gamma_1(t-x)dF(x). \tag{23}$$

For $i\geq 1$, we obtain

$$\Gamma_i(t) = \Theta_i(t) + \sum_{k=1}^{\infty}\sum_{m=1}^{N-1}\int_0^t\int_0^{t-x}\int_0^{t-x-y}\Gamma_m(t-x-y)\frac{(\lambda z)^m}{m!}e^{-\lambda(y+z)}dV(z)dV^{(k-1)}(y)d\tilde{B}^{(i)}(x)$$

$$+\sum_{k=1}^{\infty}\int_0^t\int_0^{t-x}\int_0^{t-x-y}\Gamma_N(t-x-y)\overline{V}(z)e^{-\lambda y}dF^{(N)}(z)dV^{(k-1)}(y)d\tilde{B}^{(i)}(x), \tag{24}$$

Where $\Theta_i(t) = P\{$ the time t is in one generalized repair time of the service station and the repair facility is being replaced, $\tilde{b}^{<i>} > t \geq 0\}$.

2) In order to compute $\Theta_i(t)$, we consider the following probability. Let

$\Lambda(t) = P\{$the time t is in one generalized repair time of the service station and the repair facility is being replaced$\}$.

Actually, the service station can be in one of two states: operating or breakdown in every server's generalized busy period, thus, the evolution course forms an alternating renewal process $\left\{\left(X_i,\tilde{Y}_i\right), i\geq 1\right\}$ (see Fig. 1). We obtain that

$$\Lambda(t) = \sum_{j=1}^{\infty}P\{\sum_{l=1}^{j-1}(X_l+\tilde{Y}_l)+X_j \leq t < \sum_{l=1}^{j}(X_l+\tilde{Y}_l), \text{the repair facility is being replaced at time } t\}$$

$$= \sum_{j=1}^{\infty}U(t)*\left[X^{(j)}(t)*\tilde{Y}^{(j-1)}(t)\right], \tag{25}$$

Where $U(t) = P\{\tilde{Y} > t \geq 0$, the repair facility is being replaced at time t $\}$.

Similarly, the repair facility can be in one of two states: normal or failure in each generalized repair time of the service station, and the evolution course also forms an alternating renewal process $\{(R_i, W_i), i \ge 1\}$. Decomposing $C(t)$ by $\tilde{Y}$, we get

$$U(t) = C(t) - \int_0^t C(t-x) d\tilde{Y}(x). \tag{26}$$

Therefore, we obtain $\Lambda(t)$ from expressions (25), (26) and Lemma 3.

On the other hand, the service station is operating at the beginning time and ending time of server's generalized busy period $\tilde{b}^{<i>}$. So does the repair facility. By the memory less of the exponential distribution, decomposition of $\Lambda(t)$ by $\tilde{b}^{<i>}$, it yields

$$\Theta_i(t) = \Lambda(t) - \int_0^t \Lambda(t-x) d\tilde{B}^{(i)}(x), i \ge 1. \tag{27}$$

Taking the Laplace transform on both sides of expressions (23)--(27), the proof is finished by Theorem 1, Lemma 3 and L'Hospital rule.

4.2. The expected replacement number of the repair facility during (0, t]

Set

$$H_i(t) = E\{\text{ the replacement number of the repair facility during (0, t]} \,|\, N(0) = i\}, \quad i = 0,1,\cdots.$$

Theorem 4. The Laplace-Stieltjes transform of $H_i(t)\left(i = 0,1,\cdots\right)$ are

$$\hat{h}_0(s) = \frac{r}{s + r - r\hat{w}(s)} \frac{\alpha\left[1 - \hat{\tilde{y}}(s)\right]\hat{f}(s)}{s + \alpha - \alpha\hat{\tilde{y}}(s)}\left[1 - \frac{\hat{\tilde{b}}(s)\Pi(s)}{1 - \hat{v}(s+\lambda) - \Delta(s)}\right], \tag{28}$$

$$\hat{h}_i(s) = \frac{r}{s + r - r\hat{w}(s)} \frac{\alpha\left[1 - \hat{\tilde{y}}(s)\right]}{s + \alpha - \alpha\hat{\tilde{y}}(s)}\left[1 - \frac{\hat{\tilde{b}}^i(s)\Pi(s)}{1 - \hat{v}(s+\lambda) - \Delta(s)}\right], i \ge 1, \tag{29}$$

And the rate of occurrence of failures of the repair facility is given by

$$H = \lim_{t\to\infty} \frac{H_i(t)}{t} = \lim_{s\to 0^+} s\hat{h}_i(s) = \begin{cases} \dfrac{\lambda\alpha r}{\mu\beta}, & \tilde{\rho} < 1 \\ \dfrac{\alpha r w}{\alpha\left(w + r\right) + w\beta}, & \tilde{\rho} \ge 1 \end{cases}, \tag{30}$$

Which is independent of the initial state $N(0) = i\left(i = 0,1,\cdots\right)$, $\Pi(s)$ and $\Delta(s)$ are determined by Theorem 1.

Proof. See Appendix A.

5. Other system performance measures

Denote by $C_{\min(N,V)}$ the busy cycle which is the time span between two consecutive starting points of the server's generalized busy period. Clearly, a busy cycle consists of a server's generalized busy period and a server's idle period. The server's idle period $I_{\min(N,V)}$ is defined as the time interval since the system becomes empty until the server starts to serve customers. Let $N_{\min(N,V)}$ be the number of customers in the system when the server's generalized busy period begins. One checks easily that

$$P\left\{N_{\min(N,V)}=n\right\}=\frac{1}{1-\hat{v}(\lambda)}\int_0^\infty \frac{(\lambda x)^n}{n!}e^{-\lambda x}dV(x), \quad n=1,2,\cdots,N-1,$$

$$P\left\{N_{\min(N,V)}=N\right\}=\frac{1}{1-\hat{v}(\lambda)}\sum_{n=N}^{\infty}\int_0^\infty \frac{(\lambda x)^n}{n!}e^{-\lambda x}dV(x).$$

Performing some algebraic manipulation, the mean of $N_{\min(N,V)}$ is

$$E[N_{\min(N,V))}]=\frac{N+\sum_{n=0}^{N-1}\int_0^\infty (n-N)\frac{(\lambda x)^n}{n!}e^{-\lambda x}dV(x)}{1-\hat{v}(\lambda)}. \tag{31}$$

Thus, the mean length of the server's generalized busy period $E[\tilde{b}_{\min(N,V)}]$ is

$$E[\tilde{b}_{\min(N,V)}]=E[\tilde{b}]E[N_{\min(N,V)}]=\frac{\tilde{\rho}}{\lambda(1-\tilde{\rho})}E[N_{\min(N,V)}], \quad \tilde{\rho}<1. \tag{32}$$

Moreover, the Poisson arrival process implies that

$$E[I_{\min(N,V)}]=\frac{E[N_{\min(N,V)}]}{\lambda}. \tag{33}$$

Finally, the mean length $E\left[C_{\min(N,V)}\right]$ of a busy cycle is given by

$$E\left[C_{\min(N,V)}\right]=E[\tilde{b}_{\min(N,V)}]+E[I_{\min(N,V)}]=\frac{E[N_{\min(N,V)}]}{\lambda(1-\tilde{\rho})}, \quad \tilde{\rho}<1. \tag{34}$$

6. Cost analysis and optimal threshold N*

The investigated model can be effectively applied to many real-world systems. For illustration, we consider a manufacturing system. Raw materials arrive to the system according to a Poisson process, while the processing time of raw materials follows an arbitrary distribution. Suppose that the operator (server) of the production machine takes additional tasks (e.g., preventive maintenance) when there is no raw material. Upon completion of each additional task, he/she returns to check the status of the system. If there are raw materials, the operator serves them immediately; otherwise, he/she takes another additional task. Moreover, the operator must terminate his/her additional task and start providing service as soon as N raw materials accumulate in the system. Otherwise, the storage overflows and the entire production system may stop. Further, the production machine may be interrupted due to its breakdowns. Once the production machine breaks down, it is repaired by a repair facility (e.g., crane, cutting machine). The repair facility is also subject to occasional random failures during the repair period owing to the environment influence and others. Once it fails, it is replaced by another new one and then proceeds to repair the broken production machine. The production resumes again when the broken production machine has been repaired. Consequently, such a practical example provides a good approximation of our model. In such a system, managers are interested in what is an appropriate threshold value N that minimizes the long-run expected cost function per unit time of the system.

In this section, based on the system performance measures, we develop a long-run expected cost function per unit time for the system under consideration, in which N is a decision variable. Our aim is to determine the optimum threshold N , say N*, so as to minimize the cost function. First, let

$c_h \equiv$ holding cost per unit time for each customer present in the system,

$c_r \equiv$ repair cost per unit time of the broken service station,

$c_1 \equiv$ depreciation cost incurred by every breakdown of the service station,

$c_p \equiv$ replacement cost per unit time of the failed repair facility,

$c_2 \equiv$ purchase cost incurred by every failure of the repair facility,

$c_3 \equiv$ setup cost per busy cycle.

Applying the definitions of the cost elements and its corresponding system performance measures, the long-run expected cost function per unit time is $\left(\tilde{\rho} < 1\right)$

$$C(N) = c_h \bar{L}_{\min(N,V)} + c_r \Phi + c_1 M + c_p \Gamma + c_2 H + c_3 \frac{1}{E\left[C_{\min(N,V)}\right]}$$

$$= c_h \left[\tilde{\rho} + \frac{\lambda^2 E[\tilde{\chi}^2]}{2(1-\tilde{\rho})} + \frac{N(N-1)\int_0^\infty F^{(N)}(t)dV(t) + \sum_{k=2}^{N-1}\int_0^\infty \frac{(\lambda t)^k}{(k-2)!} e^{-\lambda t} dV(t)}{2\sum_{m=1}^{N}\int_0^\infty F^{(m)}(t)dV(t)} \right]$$

$$+ c_r \frac{\lambda\alpha\left(w+r\right)}{\mu\beta w} + c_1 \frac{\lambda\alpha}{\mu} + c_p \frac{\lambda\alpha r}{\mu\beta w} + c_2 \frac{\lambda\alpha r}{\mu\beta}$$

$$+ \frac{c_3 \lambda\left(1-\tilde{\rho}\right)\left[1-\hat{v}(\lambda)\right]}{N + \sum_{n=0}^{N-1}\int_0^\infty \left(n-N\right)\frac{\left(\lambda x\right)^n}{n!} e^{-\lambda x} dV(x)}, \qquad (35)$$

Where the mean number of customers $\overline{L}_{\min(N,V)}$ in the system has been obtained in our other research work, $E\left[\tilde{\chi}^2\right]=\frac{d^2}{ds^2}\left[\hat{\tilde{g}}(s)\right]\Big|_{s=0}$.

Would have been a hard task to derive analytic result for the optimal value N* because the cost function is highly non-linear and complex. In spite of that, since N is a discrete variable, we can use direct substitution of successive values of N into the cost function until the minimum value of $C(N)$, say $C(N^*)$, is achieved. All the calculations have been done on Matlab Software and all the dates are provided here in four decimal places.

Example 1. First, the distributions of the random times involved in the system are given as follows:

(i) Service time of each customer: $G(t)=1-\exp(-\mu t)$,

(ii) Repair time of the broken service station: $Y(t)=1-\exp(-\beta t)$,

(iii) Replacement time of the failed repair facility: $W(t)=1-\exp(-wt)$,

(iv) Server's vacation time: $V(t)=\begin{cases}0, t<T\\ 1, t\geq T\end{cases}$.

Substituting these distributions into $C(N)$ and after some manipulations, it yields

$$C(N)=c_h\left\{\frac{\lambda[\beta w+\alpha(w+r)]}{\mu\beta w}+\frac{\lambda^2\left[\left(\beta w+\alpha(w+r)\right)^2+\alpha\mu\left(w+r\right)^2+\alpha\mu\beta r\right]}{\mu\beta w[\mu\beta w-\lambda\beta w-\lambda\alpha(w+r)]}\right.$$

$$\left.+\frac{N(N-1)F^{(N)}(T)+e^{-\lambda T}\sum_{k=2}^{N-1}\frac{(\lambda T)^k}{(k-2)!}}{2\sum_{m=1}^{N}F^{(m)}(T)}\right\}+c_r\frac{\lambda\alpha(w+r)}{\mu\beta w}+c_1\frac{\lambda\alpha}{\mu}$$

$$+c_p\frac{\lambda\alpha r}{\mu\beta w}+c_2\frac{\lambda\alpha r}{\mu\beta}+c_3\frac{\left(1-e^{-\lambda T}\right)\left[\lambda\mu w\beta-\lambda^2 w\beta-\lambda^2\alpha(w+r)\right]}{\mu w\beta\left[N+e^{-\lambda T}\sum_{n=0}^{N-1}(n-N)\frac{(\lambda T)^n}{n!}\right]}.$$

For convenience, we select $\lambda=0.75$, $\mu=3.0$, $\alpha=0.36$, $\beta=4.5$, $r=0.2$, $w=5.5$, $T=25$, $c_h=20$, $c_r=45$, $c_p=75$, $c_1=180$, $c_2=260$, and $c_3=380$. We compute $\tilde{\rho}=0.2707<1$. Substituting these parameters into $C(N)$, the numerical results for different threshold values of N are displayed in Table 1 and Fig. 2. From the computational results, $C(N^*) = C(5)$ =107.3214 is the minimum of the long-run expected cost per unit time. Therefore, once the number of customers waiting in the queue reaches 5, the server should interrupt his/her vacation and serve customers.

Table 1. The long-run expected cost per unit time for different values of N.

N	C(N)	N	C(N)	N	C(N)	N	C(N)	N	C(N)
1	233.5955	8	121.7215	15	176.7547	22	215.1439	29	223.8394
2	139.6741	9	128.8096	16	184.2132	23	217.7439	30	224.0442
3	115.0337	10	136.4388	17	191.1444	24	219.7364	31	224.1696
4	107.7135	11	144.4168	18	197.4378	25	221.2153	32	224.2440
5	**107.3214**	12	152.5848	19	203.0119	26	222.2781	33	224.2869
6	110.3928	13	160.7951	20	207.8207	27	223.0172	34	224.3108
7	115.4417	14	168.9009	21	211.8556	28	223.5149	35	224.3239

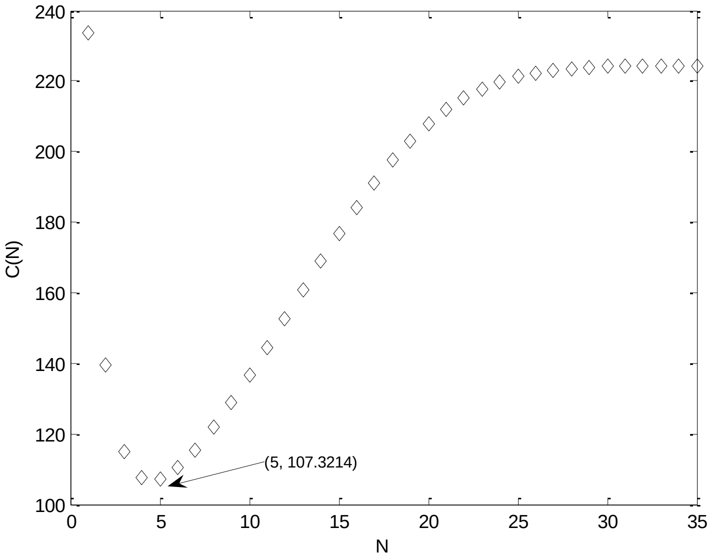

Figure 2. The plots of C(N) for different values of N.

Example 2. In this example, the distributions of the service time, repair time and replacement time are exactly the same as in Example 1, and server's vacation time is $V(t) = 1 - \exp(-vt)$. Substituting these distributions into $C(N)$ and after some calculations, it follows that

$$C(N) = c_h \left\{ \frac{\lambda[\beta w + \alpha(w+r)]}{\mu\beta w} + \frac{\lambda^2\left[(\beta w + \alpha(w+r))^2 + \alpha\mu(w+r)^2 + \alpha\mu\beta r\right]}{\mu\beta w[\mu\beta w - \lambda\beta w - \lambda\alpha(w+r)]} \right.$$

$$+\left[\frac{\lambda}{v}-\frac{N\lambda^N}{(v+\lambda)^N-\lambda^N}\right]\Bigg\}+c_r\frac{\lambda\alpha(w+r)}{\mu\beta w}+c_1\frac{\lambda\alpha}{\mu}+c_p\frac{\lambda\alpha r}{\mu\beta w}+c_2\frac{\lambda\alpha r}{\mu\beta}$$

$$+c_3\frac{v(\lambda+v)^{N-1}\left[\lambda\mu w\beta-\lambda^2 w\beta-\lambda^2\alpha(w+r)\right]}{\mu w\beta\left[(\lambda+v)^N-\lambda^N\right]}.$$

For convenience, we choose $\lambda=0.8$, $\mu=2.0$, $\alpha=0.4$, $\beta=3.0$, $r=0.2$, $w=4.5$, $v=0.25$, $c_h=40$, $c_r=55$, $c_p=90$, $c_1=160$, $c_2=240$, and $c_3=350$. We compute $\tilde{\rho}=0.4557<1$. Substituting these parameters into $C(N)$, the numerical results for different threshold values of N are shown in Table 2 and Fig. 3. From the computational results, $C(N^*)=C(3)$ =163.9966 is the minimum of the long-run expected cost per unit time. Therefore, the server should interrupt his/her vacation and start providing service as soon as 3 customers accumulate in the system.

Table 2. The long-run expected cost per unit time for different values of N.

N	C(N)	N	C(N)	N	C(N)	N	C(N)	N	C(N)
1	218.5005	8	194.0399	15	220.67927	22	228.2504	29	229.9614
2	169.8938	9	199.7243	16	222.4976	23	228.6823	30	230.0505
3	**163.9966**	10	204.7166	17	223.9970	24	229.0294	31	230.1212
4	167.5068	11	209.0358	18	225.2275	25	229.3076	32	230.1771
5	173.8316	12	212.7294	19	226.2329	26	229.5301	33	230.2213
6	180.8434	13	215.8579	20	227.0510	27	229.7076	34	230.2562
7	187.6989	14	218.4864	21	227.7143	28	229.8489	35	230.2837

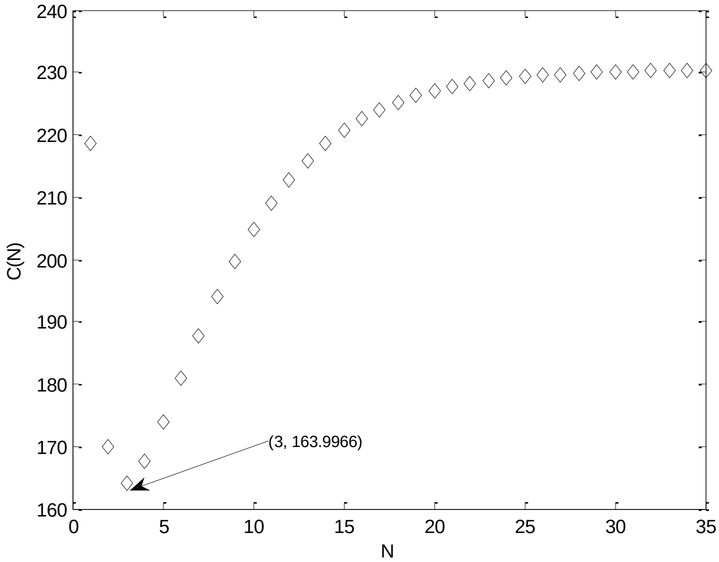

Figure 3. The plots of C(N) for different values of N.

7. Conclusion

In this paper, we considered the M/G/1 repairable queue with multiple vacations, min(N,V)-policy, unreliable service station and unreliable repair facility that has potential applications in modeling the industrial systems, the manufacturing systems and others. Applying the probability decomposition technique, various system performance measures are calculated. We then developed the long-run expected cost function per unit time of the system. In addition, using the direct search method, the optimum threshold N^* and the minimum cost $C(N^*)$ are numerically determined. It should be pointed out that our model is useful to managers who design a system with economic management. For future research, an interesting extension is to consider the Geom/G/1 repairable system with multiple vacations and N-policy (T-policy, D-policy, etc.).

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Appendix A. The proof of Theorem 4.

Proof. 1) For $i=0$, we have that

$$H_0(t)=\int_0^t H_1(t-x)dF(x)\,. \tag{A1}$$

For $i=1,2,\cdots$,

$$H_i(t)=J_i(t)+N_i(t)+\sum_{k=1}^{\infty}\sum_{m=1}^{N-1}\int_0^t\int_0^{t-x}\int_0^{t-x-y}H_m(t-x-y)\frac{(\lambda z)^m}{m!}e^{-\lambda(y+z)}dV(z)dV^{(k-1)}(y)d\tilde{B}^{(i)}(x)$$

$$+\sum_{k=1}^{\infty}\int_0^t\int_0^{t-x}\int_0^{t-x-y}H_N(t-x-y)\overline{V}(z)e^{-\lambda y}dF^{(N)}(z)dV^{(k-1)}(y)d\tilde{B}^{(i)}(x)\,, \tag{A2}$$

Where $J_i(t) = E\{$ the replacement number of the repair facility during (0, t], $\tilde{b}^{<i>} > t \geq 0\}$,

$N_i(t) = E\{$ the replacement number of the repair facility during $(0, \tilde{b}^{<i>}]$, $\tilde{b}^{<i>} \leq t\}$.

2) To compute $J_i(t)$ and $N_i(t)$, we first derive the following probability. Let

$Z(t) = E\{$ the replacement number of the repair facility during (0, t] within the alternating renewal process $\left\{\left(X_k, \tilde{Y}_k\right), k \geq 1\right\}\}$.

Because that the repair facility fails only within the generalized repair time of the service station. We have the following relationship

$$Z(t) = \int_0^t T(t-x) dP\{X \leq x\}, \quad \text{(A3)}$$

Where $T(t) = E\{$ the replacement number of the repair facility during (0, t] within the alternating renewal process $\left\{\left(\tilde{Y}_k, X_k\right), k \geq 1\right\}\}$.

Decomposition of $T(t)$ by $\tilde{Y}_1$, we know that

$$T(t) = \Lambda(t) + \Upsilon(t) + \int_0^t Z(t-x) d\tilde{Y}(x), \quad \text{(A4)}$$

Where $\Lambda(t) = E\{$the replacement number of the repair facility during $(0, t]$, $\tilde{Y}_1 > t \geq 0\}$,

$\Upsilon(t) = E\{$ the replacement number of the repair facility during $(0, \tilde{Y}_1]$, $\tilde{Y}_1 \leq t\}$.

Decompose $D(t)$ by $\tilde{Y}$, similar to expression (26), we have

$$\Lambda(t) + \Upsilon(t) = D(t) - D(t) * \tilde{Y}(t), \quad \text{(A5)}$$

Thus, we get $Z(t)$ from expressions (A3), (A4) and (A5).

Moreover, decompose $Z(t)$ by server's generalized busy period $\tilde{b}^{<i>}$, we obtain that

$$J_i(t) + N_i(t) = Z(t) - \int_0^t Z(t-x) d\tilde{B}^{(i)}(x), \quad i \geq 1. \quad \text{(A6)}$$

3) Taking the Laplace-Stieltjes transform of the expressions (A1)-(A6), and employing the L'Hospital rule, this proof is completed.

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