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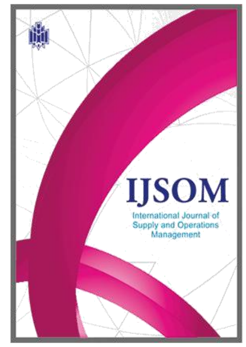

Advanced Analysis on the Behavior of PROMETHEE Methods under Independence Property and Some Derived Versions

Zhor Chergui [a,b*]

[a] *Department of Mechanics, National Higher School of Advanced Technologies (ENSTA), Algiers, Algeria*
[b] *Decision Analysis and Statistics Group, University polytechnic of Madrid, Madrid, Spain*

Abstract

This paper investigates the sensitivity of the Multicriteria decision making methods: PROMETHEE family, based on four different versions of the independence property. The analysis focuses on the robustness of each binary relation between alternatives, i.e. indifference, preference and incomparability in the overall ranking. From this optic, relevant mathematical rules and conditions are established to ensure that PROMETHEE methods maintain their initial ranking. In case of variation, a post-optimality study is proposed to anticipate the new relation without requiring a reconsideration of the method. In addition, stability intervals for each specific relation are extracted accordingly. The effectiveness of the results is demonstrated through numerical examples, two of which come from real life cases. The first deals with a project to rank alternatives for the selection of electric buses in Turkey, while the second addresses a decision-making problem linked to scheduling in a mechanical workshop for track construction in Algeria. Finally, a statistical analysis is established to compare the impact of different versions of independence using PROMETHEE I and II.

Keywords: Multicriteria Decision Making methods, PROMETHEE methods, sensitivity analysis, derived independence properties, stability intervals, simulation study.

1. Introduction

Multi-Criteria Decision Making (MCDM) or Multi-Criteria Decision Analysis (MCDA), is one of the most accurate approaches in decision-making. Since its appearance, several empirical and theoretical studies were undertaken in order to examine the mathematical modelling capability of this approach (these methods) to resolve optimization problems. The main objective was to provide an efficient framework allowing to structure decision-making problems and generate preferences from alternatives. Statistically, no one can ignore the efficiency of this approach in various fields. Indeed, MCDA methods have gained the satisfaction of several Decision Makers (DMs) along decades of use.

MCDM includes different methods that differ from each other in different aspects: the complexity level of algorithms, the weighting methods for criteria, the way of representing preferences and evaluation criteria, uncertain data

*Corresponding author email address: zhor.chergui@ensta.edu.dz
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possibility, and finally data aggregation type (cf. Taherdoost and Madanchian, 2023; Chergui, and Jiménez-Martín, 2024a; Al-Baldawi, 2024). Based on the number of alternatives, Hwang and Yoon, divided MCDM problems into two main categories. Therefore, MCDM problems are categorized into two general sub-categories: multi-attribute decision making (MADM) and multi-objective decision making (MODM). The sub-groups can be also named innumerable (with infinite admissible answers) and numerable (with finite admissible answers) (Hwang and Yoon, 1981). More explicitly, MODM focuses on continuous decision spaces with an infinite number of alternatives and is also known as continuous problems of decision making, while, MADM, which is also known as discrete problems, concentrates on problems with well-defined and finite number of decision alternatives. It is very important to bear in mind that, without loss of generality, the borders between MCDM and MADM are extensively confused, several studies tried to separate them (cf. Taherdoost and Madanchian, 2023), nevertheless, the common vision still currently unchanged.

Still regarding the classification context, a recent categorization was introduced by (Zopounidis, 2002), it consists of classifying methods regarding to their ranking procedures instead of their number of solutions as aforementioned. Four main categories are distinguished: Outranking relations, Utility functions, Discriminant function and Function free models. Hereafter, we consider the classification concept introduced by Hwang & al. (Hwang and Yoon, 1981), we focus in particular on MADM methodology and its structure.

When considering a decision rule that relies on an aggregation function within the framework of Arrow's theorem (see Abbas and Chergui, 2017; Arrow, 1963), it becomes impossible to simultaneously satisfy certain mathematical properties. Previously, Arrow (Arrow, 1963), has demonstrated that no method can fulfill the criteria of Non-dictatorship, Unanimity, Universality, Transitivity, and Independence simultaneously (refer to Arrow, 1963; Bouyssou et al., 2005; Roy and Figueira, 2009). Indeed, to establish a multicriteria model taking into account these properties, analyst should select the most relevant ones. Obviously, it is not convenient to discard the principles of Unanimity, Universality, and Non-dictatorship. Consequently, the choice is already limited to either Transitivity or Independence. In other words, aside from the three primary properties, it is not feasible to create an ordinal method that simultaneously ensures both transitivity and independence, this topic had been extensively explored in the literature (see Abbas and Chergui, 2017; Bouyssou et al., 2005; Roy and Bouyssou, 1993; Roy and Figueira, 2009; Wang and Triantaphyllou, 2006). Furthermore, detailed analysis has revealed the impossibility of identifying or defining MCDM methods that fulfill certain derived versions of independence (see Bouyssou et al., 2005; Abbas and Chergui, 2017).

The PROMETHEE family is part of Multi-Criteria Decision-Making (MCDM) methods that do not satisfy the Independence property. As noted in several studies (cf. Abbas and Chergui, 2019; Brans and Mareschal, 2001; Doan and De Smet, 2018; Mareschal, De Smet, and Nemery, 2008; Munier and Jiménez-Sàez, 2019; De Smet, 2016; Dejaegere and De Smet, 2022; Xianliang and Yunfei, 2024), the data employed in these methods influences significantly the stability of the original ranking. More explicitly, the modification or the removal of certain components or/and parameters of the decision problem after optimization, such as alternatives or criteria, is not without impact (cf. Bozôki, 2011; De Smet, 2016; Genc, 2014; Lazim and Alireza, 2018; Keyser and Peeters, 1996; Wolters and Mareschal, 1995).

In this paper, we examine the sensitivity of PROMETHEE I to four distinct versions of independence. We introduce, for the first time in the literature, certain rules and mathematical conditions insuring/eliciting the verification of these versions. Furthermore, we delineate the stability intervals within which PROMETHEE I remains robust against any data alterations. Also, for each scenario, theoretical demonstrations are provided as necessary, along with illustrative examples, some of them are derived from real-life situations.

Contrarily of the aforementioned papers (cf. Mareschal, De Smet and Nemery, 2008; Kabassi and A. Martinis, 2021; Dejaegere and De Smet, 2022; Xianliang and Yunfei, 2024), we focus on the sensitivity of PROMETHEE I instead of PROMETHEE II which is the object of most of sensitivity paper in the literature. Indeed, PROMETHEE I is regarded to be more complicated in analysis than PROMETHEE II. The presence of three binary relations (Preference, Indifference and Incomparability) as well as the comparison of two flows, makes the sensitivity study more thoroughness. However, the obtained results could be easily adjusted to deal with PROMETHEE II case. On the other hand, the large majority of existing papers introducing sensitivity analysis of PROMETHEE family focuses on weight changes and its impact on the overall ranking the reason which make from this contribution an original add value to the current findings.

This paper is organized into five principal sections. It begins with an Introduction, which provides a review of previous contributions in the field and outlines the objectives of the current study. Subsequently, we present the second section, on some generalities and notations, in which we discuss the methodology of the PROMETHEE methods. In Section 3, we detail four versions of the Independence and further reviews on the subject. Section 4, constitutes the most critical part of this research paper, it is dedicated to the primary findings regarding the sensitivity of PROMETHEE I and their stability intervals as well as illustrative examples. In Section 5, we conduct a statistical analysis comparing PROMETHEE I to PROMETHEE II under the considered independence hypotheses. The paper concludes with a Conclusion and some perspectives.

2. Some generalities and notations on PROMETHEE Methods

As previously mentioned, this paper undertakes a sensitivity analysis of the PROMETHEE methods, highlighting the objective of identifying the conditions and stability intervals under which certain modifications (see Section 3), do not alter the initial ranking of alternatives. Specifically, it seeks to ascertain the conditions and intervals in which the PROMETHEE I method remains stable when faced to this kind of modifications.

To this end, we introduce specific mathematical notations that will be referenced throughout this document. Additionally, the procedural steps involved in the PROMETHEE methods are reviewed in this section.

Consider a decision problem with a set of actions $X = \{x_1, x_2, \dots, x_m\}$ (the subject of the decision) and a criteria family F of cardinality n . For each specific criterion in F, we define a numerical function within the set of real numbers, such that: $g_j(x_i) = x_{ij}, i = \overline{1,m},\ j = \overline{1,n}$, representing the evaluations of the alternatives (actions). In addition, for each criterion, a weight w_j is assigned, which increases with the importance of the related criterion.

Furthermore, let be x and y two arbitrary actions in X. Consider also another action z, distinct from x and y, which can take various positions in the overall ranking (best alternative, worst alternative, or other).

It is important to note that the application of PROMETHEE necessitates the introduction of generalized criteria (cf. Brans and Mareschal (2001), Lazim and Alireza (2018), Seddiki and Boateng (2016)). Consequently, for each criterion j of the decision problem, a generalized criterion of the same type is associated. It is noteworthy that several types of generalized criteria have been defined in the literature, categorized into two major groups: qualitative and quantitative generalized criteria. For further literature on this subject, refer to Brans and Mareschal (2001). In our research, we examine the two most commonly used types: True criteria (Usual criterion) and Quasi criteria (U-shape Criterion).

The pairwise comparison (partial ranking of alternatives) is determined using the following preferences index:

$$\pi(x,y) = \Sigma_{j=1}^{n} w_j p_j(x,y) \tag{1}$$

where $p_j(x,y)$ is a function which measures the preference degree of x compared to y with regards to the criterion j. Its value is determined by the generalized criterion assigned to the corresponding criterion.

To construct the overall ranking of alternatives, it is necessary to compute both the outgoing flow:

$$\Phi^{+}(x) = \frac{1}{|X|-1} \sum_{y \in X \backslash \{x\}} \pi(x, y) \tag{2}$$

As well as the incoming flow:

$$\Phi^{-}(x) = \frac{1}{|X|-1} \sum_{y \in X \backslash \{x\}} \pi(y, x) \tag{3}$$

This step should be iterated as many times as the number of alternatives in the decision problem.

PROMETHEE I, the subject of our study, allows only to obtain a partial ranking, so x outranks y, if and only if:

$$\Phi^{+}(x) \geq \Phi^{+}(y) \ \ and \ \ \Phi^{+}(x) \leq \Phi^{+}(y) \tag{4}$$

with at least one strict inequality.

In some cases, the comparisons (4) may create Incomparability relations or what is commonly known by partial ranking.

The results of this paper could be subject to an easy extending to deal with PROMETHEE II. Remember that, this latest is a derived version of PROMETHEE I allowing to determine a total ranking of alternatives (possible comparison between each couple of alternatives).

Using the outgoing and incoming flow introduced in steps above, x outranks y, if and only if,

$$\Phi(x) > \Phi(y) \tag{5}$$

where $\Phi(x) = \Phi^{+}(x) - \Phi^{-}(x)$ and $\Phi(y) = \Phi^{+}(y) - \Phi^{-}(y)$,

Evidently, analysing the sensitivity of PROMETHEE I is more complex to perform than with PROMETHEE II. Therefore, it is more interesting to focus solely on the analysis of the former.

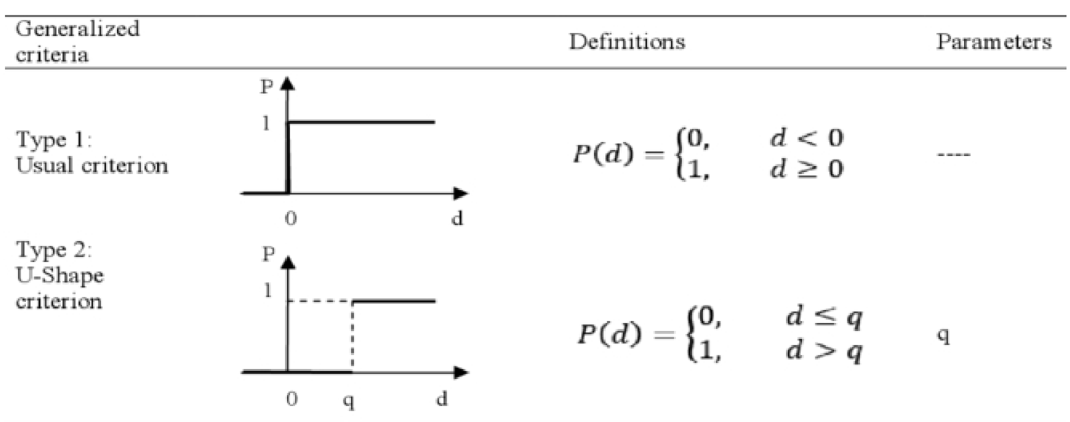


Figure 1. Generalized criteria considered

3. Independence property: some derivative versions

In decision making it is capital to verify certain mathematical properties when creating MCDM procedures. Commonly, the following properties were the most required rules in order to define an effective method: Non-dictatorship, Unanimity, Universality, Transitivity and Independence (cf. Arrow, 1963; Bouyssou and al. 2005; Roy and Figueira, 2009). Nevertheless, in the beginning of the sixtieth, it was proven that the simultaneous verification of these properties is not possible, it is the famous finding of ARROW (cf. Arrow, 1963; Bouyssou and al. 2005; Roy and Figueira, 2009). For this reason, researchers already try to weaken some of these requirements by violating the less desired properties. Independence property is the less verified version by the most of multicriteria methods. PROMETHEE, as some other MCDM methods uses the all of all alternatives ranking which affect directly the verification of this property (cf. Bouyssou and al. 2005). Depending on the position of alternatives in the overall ranking and the eventual action undertaken on them (delete, change, add), several versions of Independence are introduced in the literature (cf. Abbas and Chergui, 2017; Vincke, 1992; Wang and Triantaphyllou, 2006). In the present study, we focus on the most appeared versions, to each one its own framework and impact on the overall ranking. Hereafter, a description of the chosen versions is provided:

Naturally, since PROMETHEE I doesn't verify the Independence property, it is previsible that it couldn't verify their derived versions as well. A numerical example can easily show the fact that in some cases altering the performance table can't affect the overall ranking (for numerical examples cf. Vincke, 1992; Abbas and Chergui, 2017). However, not any change remains the overall ranking unchanged. This kind of information seems to be very important to the decision maker in order to know the limits of her(his) actions. In the following section, we study the behaviour of PROMETHEE I under the above Independence versions. We carry out a sensitivity analysis of this method where the purpose is to predict the impact of an eventual action on the overall ranking of alternatives.

Version 1: The principle underlying the first version consists in substituting a suboptimal action z with another z' which is less good. PROMETHEE I verifies this version if, and only if, the initial overall ranking remains stable under the described modification. In other words, whatever the components of the new alternative z' the initial overall ranking should be preserved. This version was part of a series of tests introduced by Triantaphyllou and al. (cf. Wang and Triantaphyllou, 2006; Abbas and Chergui, 2017; Vincke, 1992;).

Version 2: {The independence of non-discriminating element} this version is hold by PROMETHEE I, if the removal of a given alternative z does not affect the initial overall ranking (cf. Vincke, 1992).

Version 3: {Independence of the best or the worst ranked element:} PROMETHEE I verifies this version if the removal of the best (or worst) alternative z does not alter the overall ranking (cf. Vincke, 1992).

Version 4: {Independence of the best or the worst set of ranked elements:} PROMETHEE I fulfills this version if the removal of a group of the best (or worst) alternatives does not change the overall ranking (cf. Vincke, 1992).

Evidently, PROMETHEE I does not satisfy the Independence property, it is predictable that it would not satisfy the derived versions either. A numerical example can readily demonstrate that, in certain cases, altering the performance table does not impact the overall ranking (for numerical examples cf. Vincke, 1992; Abbas and Chergui, 2017). However, not every change leaves the overall ranking unaffected. A such information is crucial for the decision-maker(s) (DM) in order to better understand the limitations of its decision problem. In the following section, we examine the behavior of PROMETHEE I under the aforementioned Independence versions. We conduct a sensitivity analysis of this method to predict the impact of potential actions on the overall ranking of alternatives.

4. PROMETHEE I and the impact of Independence versions: analysis study and conclusions

Several sensitivity studies concerning the weights evolution and its impact on the results of PROMETHEE methods were carried out in the literature (Vahabzadeh and Khamseh, 2024). Yet, none of them investigates theoretically the robustness of PROMETHEE I neither the relative confidence intervals when altering in the performance table.

In this section, we examine the performance of PROMETHEE I under the four derived versions of Independence as described in the preceding section:

Version 1

Consider replacing a non-optimal action z with another one less good, denoted by z', such that: $\Delta_j = z_j - z'_j \geq 0$, $j = \overline{1, n}$, with at least one strict inequality.

A thorough analysis of the preference computational steps of PROMETHEE I , it is easy to observe that if the new alternative z' alters at least one value $p_j(x, z')$ (resp. $p_j(z', x)$) in the preference indices $\pi(x, z')$ (resp. $\pi(z', x)$), the initial ranking may be affected. This is because, as demonstrated in the mathematical formulae of the outgoing flow (2) and (3), all alternatives in the problem are considered when computing the flow.

In the same context, an indifference relation between two alternatives x and y is maintained if and only if:

$$D_x^+ = D_y^+ \text{ and } D_x^- = D_y^- \quad (6)$$

Where, $D_x^+ = \pi(x, z') - \pi(x, z)$, and $D_x^- = \pi(z, x) - \pi(z', x)$. For a complete proof (cf. Abbas and Chergui, 2017).

Conversely, in the case of a preference relation:

Proposition 1:

x is preferred to y in the initial ranking, this relation is preserved, if and only if, the following inequalities are verified:

$$D_x^+ \geq D_y^+ - (|X| - 1)[\Phi^+(x) - \Phi^+(y)] \quad (7)$$

And

$$D_x^- \leq D_y^- - (|X| - 1)[\Phi^-(y) - \Phi^-(x)] \quad (8)$$

with at least one strict inequality.

Proof: To prove these inequalities, we assume that x is preferred to y in the initial ranking, which provides: $\Phi^+(x) > \Phi^+(y)$.

Let be, $D_x^+ \geq D_y^+ - (|X| - 1)[\Phi^+(x) - \Phi^+(y)]$,

Then, $\pi(x, z') - \pi(x, z) \geq \pi(y, z') - \pi(y, z) - (|X| - 1)[\Phi^+(x) - \Phi^+(y)]$;

This implies that: $\Phi_{z'}^+(x) \geq \Phi_{z'}^+(y)$.

We proceed in the same manner to demonstrate that $\Phi_{z'}^-(x) \leq \Phi_{z'}^-(y)$, hence x is preferred to y in the new ranking.

Now, we consider that x is preferred to y in the new ranking, this implies that $\Phi_{z'}^+(x) \geq \Phi_{z'}^+(y)$,

However, $\Phi_{z'}^+(x) + \frac{1}{|X|-1}\left(\pi(x, z) - \pi(x, z)\right) \geq \Phi_{z'}^+(y) + \frac{1}{|X|-1}\left(\pi(y, z) - \pi(y, z)\right)$;

Then, $(\pi(x, z') - \pi(x, z)) \geq (\pi(y, z') - \pi(y, z)) - (|X| - 1)[\Phi^+(x) - \Phi^+(y)]$;

Hence, $D_x^+ \geq D_y^+ - (|X| - 1)[\Phi^+(x) - \Phi^+(y)]$.

By analogy, we show the inequality (8) of the Proposition 1.

By analogy, we prove the second inequality of the formula.

Proposition 2:

The stability intervals to maintain a preference relation in the overall ranking after altering, are as follow:

$$D_x^+ \in \left[D_y^+ - (|X| - 1)[\Phi^+(x) - \Phi^+(y)], 1\right] \quad (9)$$

and

$$D_x^- \in \left[0, D_y^- - (|X| - 1)[\Phi^-(y) - \Phi^-(x)]\right] \quad (10)$$

Corollary: The stability intervals belong to the defined interval: $[0,1]$.

The impact on Incomparability relation, Proposition 3 shows the relative conditions:

Proposition 3:

Assume that, x is incomparable to y in the initial ranking, this relation is maintained, if and only if, the following inequalities are hold:

$$D_x^+ > (resp. <) D_y^+ - (|X| - 1)[\Phi^+(\mathrm{x}) - \Phi^+(\mathrm{y})] \quad (11)$$

And

$$D_x^- > (resp. <) D_y^- - (|X| - 1)[\Phi^-(\mathrm{y}) - \Phi^-(\mathrm{x})] \quad (12)$$

Remark: It is possible to prove the Incomparability and Indifference cases for this version and for the subsequent versions using the same steps as in the preference relation.

Through these recent results, it is evident that Indifference is the most sensitive relation to this change. However, as mentioned earlier, changes that satisfy the above conditions do not affect the overall ranking.

Numerical example:

The numerical example presented in Table 1 provides a demonstration to Proposition 2, related to the preference relation. In this decision problem, the case of altering the overall ranking after the introduction of a new alternative in place of a non-optimal alternative is carefully discussed.

Consider the relative performance matrix including four criteria and three alternatives. The evaluations are detailed in table 1. as follows:

Table 1. Performance matrix.

	C_1	C_2	C_3	C_4
A_1	3	2	1	2
A_2	2	1	2	1
A_3	3	2	3	1

In this example, and in all subsequent examples unless otherwise specified, the criteria are to be maximized, and the weights are considered equal.

The resolution using PROMETHEE I yields the overall ranking: $A_3 \succ A_1 \succ A_2$, where "$\succ$" denotes outranking (preference relation). This ranking results from the comparison of the flows between actions. The numerical results are illustrated in the following table:

Table 2. Flows and Global preferences.

	A_1	A_2	A_3	$\Phi^+(A_.)$
A_1	-	**3/4**	1/4	1/2
A_2	**1/4**	-	**0**	1/8
A_3	1/4	**3/4**	-	1/2
$\Phi^-(A_.)$	1/4	3/4	1/8	-

The values in the cells represent the global preferences quantities.

The substitution of the alternative A_2 by a new less good alternative $A'_2 = (2,1,0,1)$, defines a new overall ranking (see Table 3), where:

Table 3. Flows and Global preferences after change.

	A_1	A_2	A_3	$\Phi^+(A_.)$
A_1	-	1	1/4	5/8
A_2	0	-	0	0
A_3	1/4	3/4	-	1/2
$\Phi^-(A_.)$	1/8	7/8	1/8	-

PROMETHEE I is sensitive to this modification. It is noteworthy that this alteration significantly affects the outgoing flow values of A_1, which is more important than the change observed in A_3. This situation can be easily predicted according to the differing values of the characteristic quantities in Table 4:

Table 4. Characteristic quantities.

D_3^+	$D_1^+ - (m-1)[\Phi^+(A_3) - \Phi^+(A_1)]$
0	1/4
D_3^-	$D_1^- - (m-1)[\Phi^-(A_3) - \Phi^-(A_1)]$
0	0

Furthermore, based on the data provided, the stability intervals for the preference relation between and are as follows:

$$D_3^+ \in [0.25, 0] \text{ and } D_3^- = 0$$

It is important to mention that, the impact on the global ranking is primarily contingent upon the type of generalized criterion employed.

Version 2:

PROMETHEE I is sensitive to the removal of a given alternative z. Indeed, The method's steps in section 2. show that the calculation of the flows uses all the set of alternatives renders the non-verification of this version of Independence an unavoidable outcome. However, as in the aforementioned case, what conditions precipitate the alteration of a given relation?

Proposition 4:

An indifference relation holds if and only if:

$$\pi(x,z) = \pi(y,z) \textit{ and } \pi(z,x) = \pi(z,y) \quad (13)$$

where x and y are indifferent in the initial ranking (before the delete of z).

Consider that x is preferred over y in the initial ranking.

Proof: Let be $\Phi^+(x) = \Phi^+(y)$ and $\Phi^-(x) = \Phi^-(y)$ (x is preferred than y), since the quantities $\pi(x,z)$ and $\pi(y,z)$ (resp. $\pi(z,x)$ and $\pi(z,y)$) are removed from the $\Phi^+(.)$ (resp. $\Phi^-(.)$) expressions, they must be equal to maintain equality between the flows.

Proposition 5:

A preference relation remains unchanged if and only if,

$$\Phi^{+}_{X\backslash z}(x) \geq \Phi^{+}_{X\backslash z}(y) \quad (14)$$

and

$$\Phi^{-}_{X\backslash z}(x) \leq \Phi^{-}_{X\backslash z}(y) \quad (15)$$

or

$$\pi(x,z) \leq \pi(y,z) - (|X|-1)[\Phi^{+}(y) - \Phi^{+}(x)] \quad (16)$$

and

$$\pi(z,x) \geq \pi(z,y) - (|X|-1)[\Phi^{-}(y) - \Phi^{-}(x)] \quad (17)$$

with at least one strict inequality.

Proof: Consider $\Phi^{+}_{X\backslash z}(x) \geq \Phi^{+}_{X\backslash z}(y)$,

Then: $\frac{1}{|X|-1}\big(\pi(x,z) - \pi(x,z)\big) + \Phi^{+}_{|X|\backslash z}(x) \geq \Phi^{+}_{|X|\backslash z}(y) + \frac{1}{|X|-1}\big(\pi(y,z) - \pi(y,z)\big)$.

Also, $\frac{1}{|X|-1}\pi(x,z) + \Phi^{+}(x) \geq \Phi^{+}(y) - \frac{1}{|X|-1}\pi(y,z)$.

This implies that, $\pi(x,z) \leq \pi(y,z) - (|X|-1)\big(\Phi^{+}(y) - \Phi^{+}(x)\big)$.

By analogy, the second inequality can be easily proven.

Proposition 6:

The stability intervals relative to a preference relation are as follow:

$$\pi(x,z) \in \big[0, \pi(y,z) - (|X|-1)[\Phi^{+}(y) - \Phi^{+}(x)]\big] \quad (18)$$

And

$$\pi(z,x) \in [\pi(z,y) - (|X|-1)[\Phi^{-}(y) - \Phi^{-}(x)], 1] \quad (19)$$

In case of change with equality in both last conditions, the preference relation becomes indifference.

Regarding the Incomparability relation, assume initially that x is incomparable to y in the overall ranking.

Proposition 7:

An incomparability relation is maintained by the removal of a given alternative z, if and only if,

$$\Phi^{+}_{X\backslash z}(x) > (resp. <)\Phi^{+}_{X\backslash z}(y) \quad (20)$$

And

$$\Phi^{-}_{X\backslash z}(x) > (resp. <)\Phi^{-}_{X\backslash z}(y) \quad (21)$$

Or

$$\pi(x,z) < (resp. >)\pi(y,z) - (|X|-1)[\Phi^{+}(y) - \Phi^{+}(x)] \quad (22)$$

And

$$\pi(z,x) < (resp. >)\pi(z,y) - (|X|-1)[\Phi^{-}(y) - \Phi^{-}(x)] \quad (23)$$

Proof: To prove Proposition 7, we proceed in the same manner as in the preference relation case.

Numerical example:

The present example shows the case of Incomparability relation. The decision problem is composed of four alternatives and three pseudo-criteria (for more information on this generalized criterion cf. Brans and Mareschal, 2001), the relative thresholds of Indifference and Preference are, q = 0.2 and p = 0.6:

Table 5. Performance matrix.

	C1	**C2**	**C3**
A_1	1	1	1
A_2	3	3	2
A_3	3	2,6	2
A_4	3	2	3

The resolution of this decision maker by PROMETHEE I results an Incomparability relation between A_2 and A_4 (for the flows values, see table 6):

Table 6. Flows and Global preferences.

	A_1	A_2	A_3	A_4	$\Phi^+(A_.)$
A_1	-	0	0	0	0
A_2	1	-	1/6	1/3	9/18
A_3	1	0	-	1/3	4/9
A_4	1	1/3	1/3	-	5/9
$\Phi^-(A_.)$	1	1/9	3/18	2/9	-

Removing the alternative A_3 from the alternatives set, alters the relation between A_2 and A_4 by an indifference relation (table 7):

Table 7. Flows and Global preferences after change.

	A_1	A_2	A_4	$\Phi^+(A_.)$
A_1	-	0	0	0
A_2	1	-	1/3	4/6
A_4	1	1/3	-	4/6
$\Phi^-(A_.)$	2/3	1/6	1/6	-

The characteristic quantities for the incomparability relation are defined (see table 8):

Table 8. Characteristic quantities

$\pi(A_2, A_3)$	$\pi(A_4, A_3) - 3[\Phi^+(A_4) - \Phi^+(A_2)]$
0	0
$\pi(A_3, A_2)$	$\pi(A_3, A_4) - 3[\Phi^-(A_4) - \Phi^-(A_2)]$
0	0

Since the inequalities of Incomparability relation are not verified for at least one of them, the Incomparability relation cannot be conserved. furthermore, remark that the quantities in the table are equal which satisfies the indifference relation between A_2 and A_4 in the new ranking.

Version 3:

In this version the sensitivity of PROMETHEE I is tested to the remove of the best (resp. worst) alternative. As version 2, the same mathematical conditions remain the same. Yet, the impact of this action is not with the same intensity. Indeed, removing the best alternative in PROMETHEE I, strengthen the preference relations than were in the initial ranking and grows the chances to the appearance of new other preference relations between the remained alternatives. The decision problem bellow illustrates the creation of a new preference relation from an initial Incomparability relation.

Numerical examples:

The decision problem is formed of four alternatives and three criteria, the relative performance matrix is as follows:

Table 9. Performance matrix.

	C1	C2	C3
A_1	1	1	3
A_2	3	3	1
A_3	3	2	1
A_4	3	2	2

In the overall ranking, A_4 and A_2 are the best alternatives and preferred to the other alternatives. A_1 and A_3 are incomparable, the computation of the flows outputs the following quantities:

Table 10. Flows and Global preferences.

	A_1	A_2	A_3	A_4	$\Phi^+(A_.)$
A_1	-	1/3	1/3	1/3	1/3
A_2	2/3	-	1/3	1/3	4/9
A_3	2/3	0	-	0	2/9
A_4	2/3	1/3	1/3	-	4/9
$\Phi^-(A_.)$	6/9	2/9	3/9	2/9	-

Removing A_2 transforms the incomparability relation between A_1 and A_3 to a preference relation. In the table 11 below, the first inequality of the incomparability condition is not checked:

Table 11. Characteristic quantities

$\pi(A_1, A_2)$	$\pi(A_3, A_2) - 3[\Phi^+(A_3) - \Phi^+(A_1)]$
1/3	1/3
$\pi(A_2, A_1)$	$\pi(A_2, A_3) - 3[\Phi^-(A_3) - \Phi^-(A_1)]$
2/3	4/3

The characteristic quantities show that the incomparability relation is not preserved, affected by the delete of A_2, one of the best alternatives. Moreover, A_3 becomes now preferred to A_1, an illustration of the strengthen of the initial relation between A_3 and A_1.

In this second example, a similar phenomenon to the one described in the first one is presented.

The decision problem is relative to a real-life scenario consisting the selection of electric buses for urban mass transportation introduced in Hamurcu and al. (cf. Hamurcu and Eren, 2020). The proposed structure of the decision problem includes six criteria:

- Speed (C1): Fastness of Electric bases.

- Passenger capacity (C2): Capacity of transportation.

- Range (C3): Given the limited range of electric vehicles, this factor is a critical specific feature. A longer range implies greater network area involvement.

- Maximum Power (C4): The electric motor capacity.

- Battery capacity (C5): The capacity of batteries.

- Charging time (C6): charging times or the batteries.

And six alternatives representing potential electric buses. The corresponding data (evaluations) is detailed in table 12:

Table 12. Characteristic quantities

	C1	**C2**	**C3**	**C4**	**C5**	**C6**
A_1	72	50	200	360	360	2
A_2	68.4	58	200	360	394	1.25
A_3	90	50	280	103	170	7
A_4	80	57	50	200	200	2
A_5	75	90	280	250	230	5
A_6	75	136	300	250	346	7

The priorities upon criteria are quantified resulting the weights vector: $w = (0.0710, 0.1196, 0.1529, 0.1014, 0.3428, 0.2123)$. In this example, the criteria are all taken as usual generalized criteria and criterion 6 ($C6$) is the only minimized criterion.

Using PROMETHEE-GAIA software to resolve this instance, the overall ranking is defined (see in figure 2).

The removal of the best alternative A_2 modifies the relation between A_1 and A_6 from incomparability to preference. From the PROMETHEE-GAIA tool, Figure 3 shows the new overall ranking:

The sensitivity conditions introduced for this version are not verified which justifies the creation of this new relation (see Table 13):

Table 13. Characteristic quantities

$\pi(A_1, A_2)$	$\pi(A_6, A_2) - 5[\Phi^+(A_6) - \Phi^+(A_1)]$
0.071	0.2455
$\pi(A_2, A_1)$	$\pi(A_2, A_6) - 5[\Phi^-(A_6) - \Phi^-(A_1)]$
0.6747	0.553

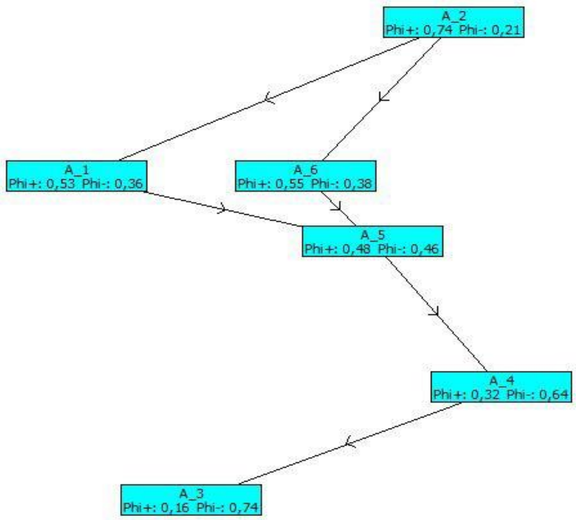


Figure 2. Global ranking.

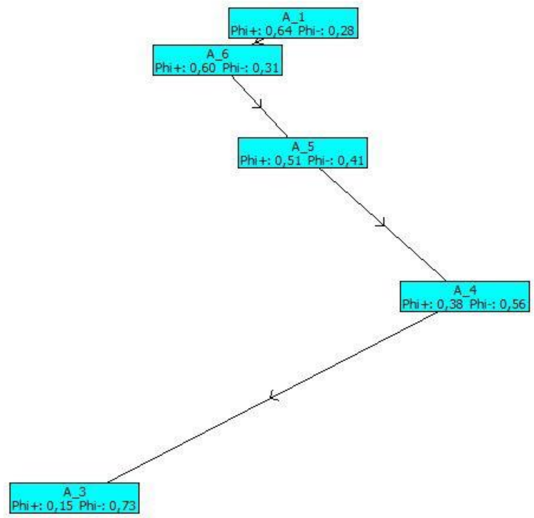


Figure 3. Global ranking after delete

In two different examples, deleting the best alternatives creates new preference relation at each case. Indeed, as it is mentioned before, delete the best alternative(s) enhance the chance to create new preference relations in the overall ranking from other weak types of relation.

Version 4:

The removal of a set of best alternatives could be seen as a generalization of the derived Independence property number 3 (Version 3). For this reason, an adaptation of the propositions in Version 3 is naturally possible.

Firstly, we denoted by B the set of the best alternatives, x and y are two indifferent alternatives in the initial ranking such that: $\{x\} \notin B$ and $\{y\} \notin B$

Proposition 8: an indifference relation is maintained after removing the set of best alternatives B, if and only if,

$$\sum_{\mathrm{z} \in B} \pi(x, \mathrm{z}) = \sum_{\mathrm{z} \in B} \pi(\mathrm{y}, \mathrm{z}) \tag{24}$$

and

$$\sum_{A_i \in B} \pi(z, \mathrm{x}) = \sum_{A_i \in B} \pi(\mathrm{z}, \mathrm{y}) \tag{25}$$

Regarding the preference relation:

Proposition 9:

Assume that: $\{x\} \notin B$ and $\{y\} \notin B$, x is preferred to y in the initial ranking (established by PROMETHEE I). This relation remains unchanged if and only if:

$$\sum_{z \in B} \pi(\mathrm{x}, \mathrm{z}) \leq \sum_{A_i \in B} \pi(\mathrm{y}, \mathrm{z}) - (|X| - 1)[\Phi^+(\mathrm{y}) - \Phi^+(\mathrm{x})] \tag{26}$$

and

$$\sum_{z\in B}\pi(\mathrm{z},\mathrm{x}) \geq \sum_{z\in B}\pi(\mathrm{z},\mathrm{y}) - (|X|-1)[\Phi^{-}(\mathrm{y}) - \Phi^{-}(\mathrm{x})] \tag{27}$$

Proposition 10:

The stability intervals relative to maintain a preference relation between two actions x and y, such that: xPy, are as follows:

$$\sum_{z\in B}\pi(\mathrm{x},\mathrm{z}) \in \left[0, \sum_{z\in B}\pi(y,\mathrm{z}) - (|X|-1)[\Phi^{+}(\mathrm{y}) - \Phi^{+}(\mathrm{x})]\right] \tag{28}$$

and

$$\sum_{z\in B}\pi(\mathrm{z},\mathrm{x}) \in \left[\sum_{z\in B}\pi(\mathrm{z},\mathrm{y}) - (|X|-1)[\Phi^{-}(\mathrm{y}) - \Phi^{-}(\mathrm{x})], |B|\right] \tag{29}$$

With regards to Incomparability relation:

Proposition 11:

Let be x and y two in compared alternatives in the global preorder, such that: $\{x\} \notin B$ and $\{y\} \notin B$. This relation remains unchanged if and only if:

$$\sum_{z\in B}\pi(\mathrm{x},\mathrm{z}) < (resp. >) \sum_{z\in B}\pi(\mathrm{y},\mathrm{z}) - (|X|-1)[\Phi^{+}(\mathrm{y}) - \Phi^{+}(\mathrm{x})] \tag{30}$$

and

$$\sum_{z\in B}\pi(\mathrm{z},\mathrm{x}) < (resp. >) \sum_{z\in B}\pi(\mathrm{z},\mathrm{y}) - (|X|-1)[\Phi^{-}(\mathrm{y}) - \Phi^{-}(\mathrm{x})] \tag{31}$$

Numerical example:

In a mechanical engineering workshop, optimizing the production time function is strongly advised. Achieving optimal scheduling is a key solution for enhancing service quality. This case examines decision-making within an Algerian truck fabrication workshop where two types of pieces of machinery, Bride 180 and Spools with different diameters, are produced. During one stage of the production process, the same drilling machine is utilized for both types. To prevent machine congestion, minimize idle time, and adhere to the deadlines for each component, a multicriteria decision function is established. This function employs three criteria of equal priority: C1 for processing time, C2 for due date, and C3 for achievement date from the previous machine.

In one operation, six alternatives (six pieces of the two mentioned types) are considered, the evaluations (by minutes) across the three criteria are detailed in Table 14.

Using PROMETHEE I, the resulting overall rankings indicate that: $A_6 \succ A_4 \succ A_3 \succ A_1 = A_2 \succ A_5$. since the machine can only handle one piece at a time, two possible rankings emerge depending to the position of A_1 over A_2. Practically speaking, the optimal solution is the one securing a minimum of Makespan with the subsequent machines.

Consider the set $B = \{A_4, A_6\}$ containing the best alternatives. By eliminating B from the performance matrix in Table 14, a new relation between the remaining alternatives is established (A_1, A_2 and A_5).

Table 14. Performances matrix

	C1	C2	C3
A_1	9.2	360	315.87
A_2	12.9	320	275.77
A_3	15.3	270	203.60
A_4	12.88	250	218.42
A_5	30	300	259.31
A_6	15.17	200	155.32

Table 15. Characteristic quantities

$\sum_{A_i \in B} \pi(A_1, A_i)$	$\sum_{A_i \in B} \pi(A_5, A_i) - (m-1)[\Phi^+(A_5) - \Phi^+(A_1)]$
$2/3$	$1/3$
$\sum_{A_i \in B} \pi(A_i, A_1)$	$\sum_{A_i \in B} \pi(A_i, A_1) - (m-1)[\Phi^+(A_5) - \Phi^+(A_1)]$
$4/3$	$5/3$

The quantities in table 15, demonstrate a change in the initial relation between A_5 and A_1. Indeed, A_5 moved from the queue to a better position where A_1 is moved back, then the new overall ranking became as follow: $A_3 \succ A_5 = A_2 \succ A_1$. This example illustrates that PROMETHEE cannot maintain its ranking when a set of good alternatives is removed.

An example of a such change influencing an incomparability relation can be shown with the scenario in version 3 taken into account the set $B = \{A_4, A_2\}$.

5. Statistical study and comparison

To study the effect of altering data on the overall ranking of alternatives using PROMETHEE I and PROMETHEE II, a statistical analysis is conducted, focusing on the impact of deleting or modifying a specific alternative z on the rankings of the other alternatives. Three distinct scenarios concerning z's position in the global ranking are examined: 1) z is the best alternative, 2) z is the worst alternative, and 3) z is neither the best nor the worst alternative.

In each scenario, a comparison is made between the original ranking and the ranking after deleting z (resp. changing it by a worst one) is established, after that, the Hit ratio which is a statistical metric is calculated. Its mechanism is to compute the number of times the original ranking is the same as the ranking after taking an action on the chosen alternative (deleting or altering).

For this purpose, a software application has been developed to generate decision-making data problems and solve them using PROMETHEE I and PROMETHEE II, taking into account the described situations. For each fixed number of alternatives (ranging from 3 to 20) and criteria (from 3 to 25), 10,000 instances of weight vectors and performance matrices are randomly generated according to a Uniform distribution between 0.1 and 0.99. (U (0.1, 0.99)). The generated data is normalized using formulas from (Chergui and Jiménez Martin, 2024(a)(b)(c)(d)). After that, each instance is solved twice by each method, before and after action and on this basis the Hit ratio is computed.

The Hit ratio represents the proportion of the 10,000 instances executed in which the corresponding method outputs the same best alternative as in the First ranking (before action) (Chergui and Jiménez Martin, 2024(a)(b)(c)(d)).

The graphs below illustrate the obtained results:

The vertical axis represents the rate of instances maintaining the same ranking as the original one (Hit ratio) and the horizontal axis indicating the number of alternatives. Modifying an alternative, regardless of its position, has a slightly lesser impact than deleting it with both methods, and the deletion or modification of the best alternative significantly influences the overall rankings more than the delete/modification of any other alternative. The deletion of the worst alternative came second in the impact intensity.

Finally, PROMETHEE II is more robust than PROMETHEE I to the changes described previously.

This comparison analysis shows clear instabilities in the behaviour of both methods. Mathematically speaking, this observation emphasis the need to develop new well-founded results illustrating the relative impact of data alteration. For this reason, the outcomes in section 4 will shorten the path to the practitioner by proposing new sensitivity rules and conditions allowing to provide clear improvements to decision support systems dealing with PROMETHEE family methods.

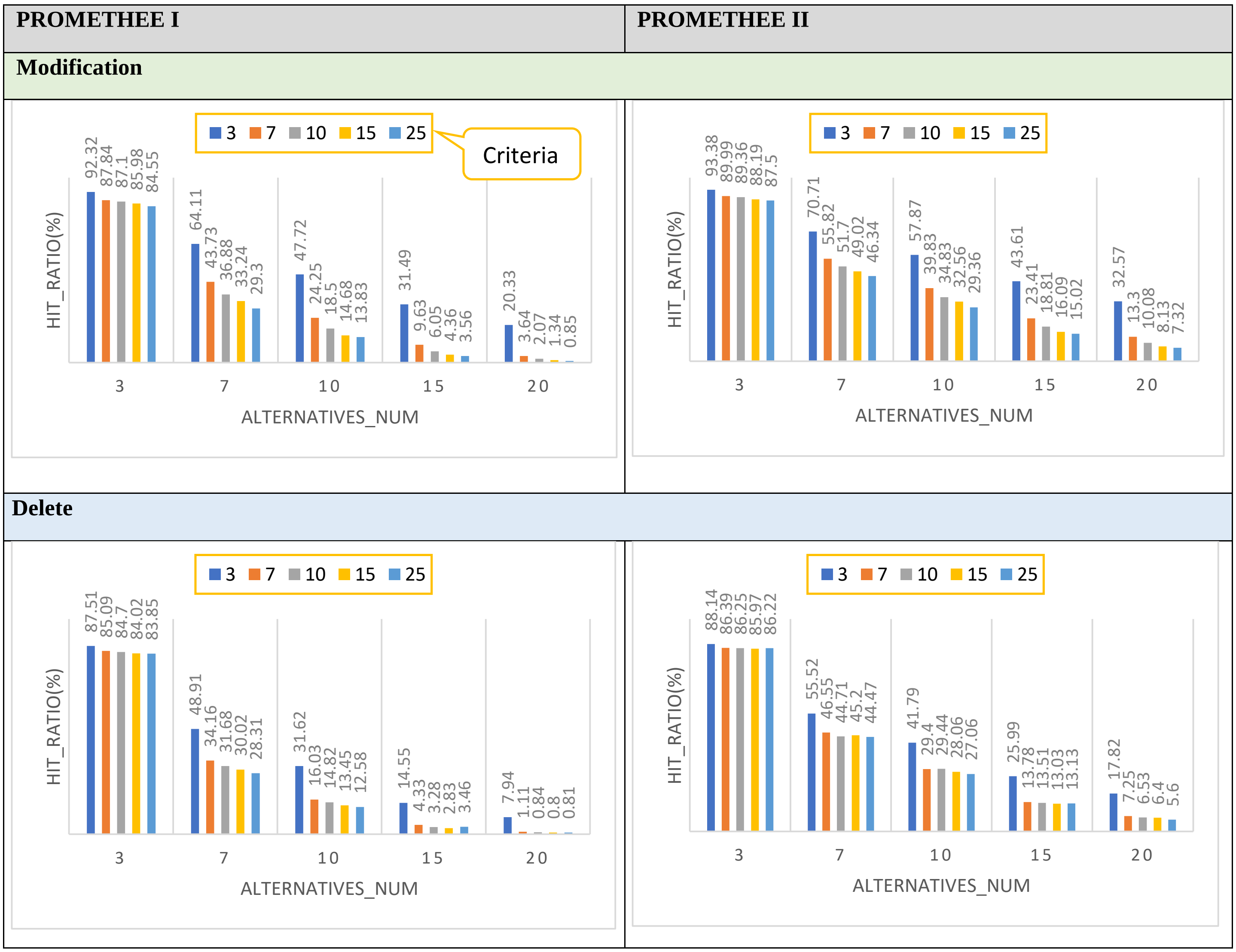
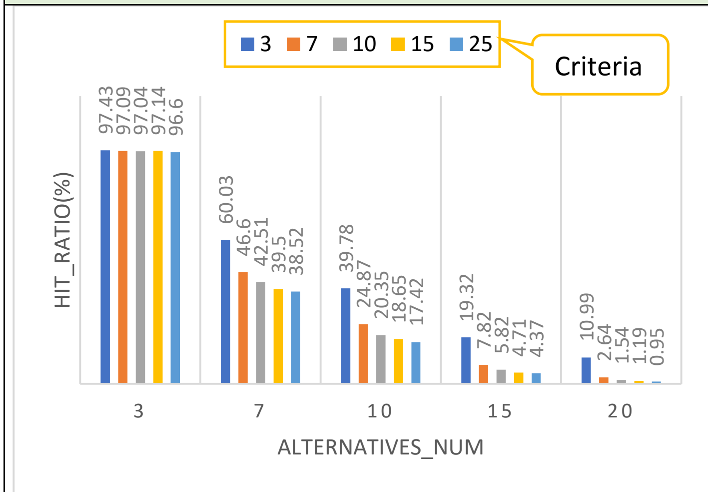
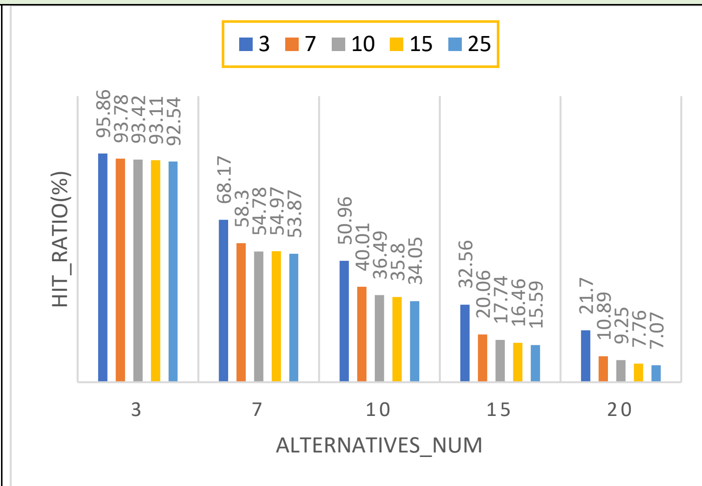


Figure 4. Comparison study: case of the worst Alternative

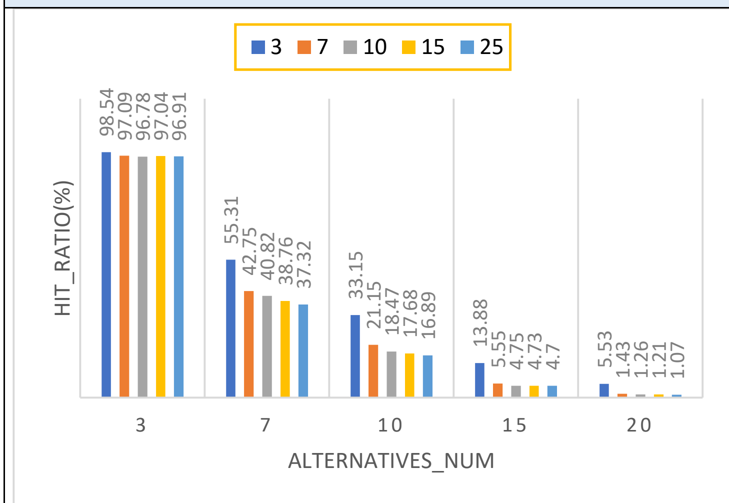
PROMETHEE I	**PROMETHEE II**

Modification





Delete



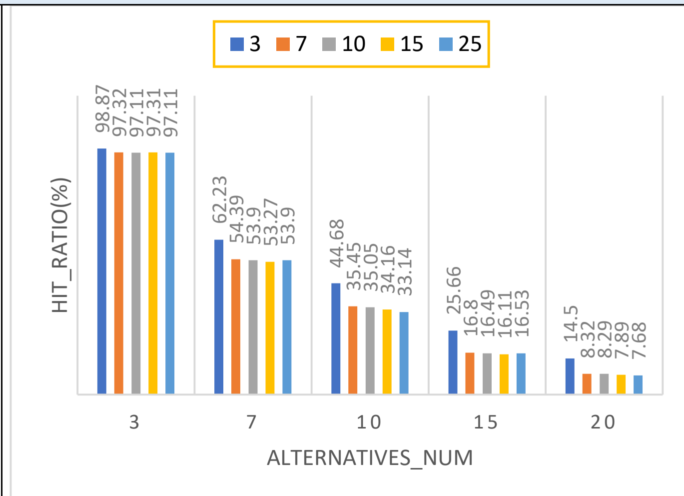


Figure 5. Comparison study: case of any Alternative

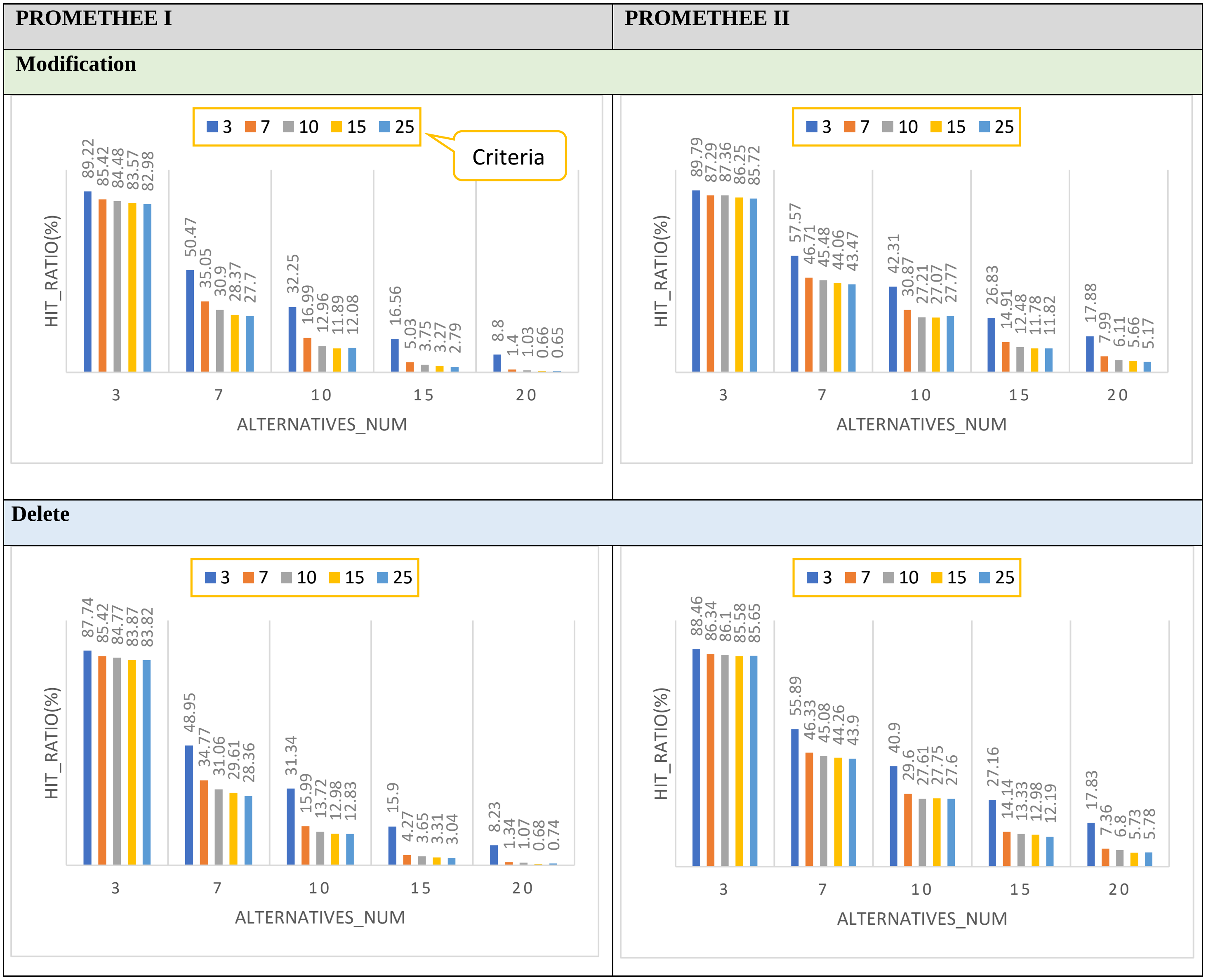


Figure 6. Comparison study: case of the best Alternative

6. Conclusion

The post optimality methodology in multicriteria decision making domain, is a key concept, enables to explore the limitations of the initial ranking and its sensitivity to an eventual modification. It also provides immediate insights into the impact of these changes while avoiding further processing of the used method.

PROMETHEE methods are recognized for their efficiency and stability over decades of application; however, a post-optimal study is strongly advised in order to understand and illustrate the features of the method using the presented data and the relation them. This paper examines the sensitivity of PROMETHEE I considering four versions of the independence property. Theoretical rules and conditions regarding the impact of data changes on the three types of logic relation in the overall ranking are introduced for the first time in the literature. Numerical examples are used to illustrate the obtained results, showing both the sensitivity of the method and its stability intervals. All types of the ranking relation are explored, and several real-life situations are discussed.

This paper aims to provide an advanced/practical interpretation of the results found in Vincke, 1992. As future directions, we propose conducting sensitivity studies on Independence and transitivity properties using PROMETHEE for fuzzy information and other MCDM methods, analyzing new sensitivity cases with various generalized criteria. Moreover, to enhance the purpose of this study, it would be beneficial to incorporate these results into a decision support system that facilitates the generation of detailed sensitivity reports for PROMETHEE I and other MCDA methods.

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