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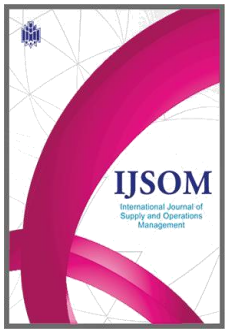

A Copula-based Maintenance Modeling for Parallel Systems with Non-self-Announcing Failures and Dependent Components

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Abstract
Based on a copula function, this paper addresses a maintenance scheduling problem for parallel systems whose components are dependent and their failures are detected only by inspections. To carry out preventive maintenance actions, the decision process is steered by the excursions of the state process $X(t)$ describing the total number of failed components up to age t. Since both the maintenance costs and the level of maintenance are driven by the inspection interval τ and the preventive replacement threshold j, using the standard renewal theory arguments, the paper aims to jointly determine both optimal inspection and optimal replacement policy to truly balance the two factors. The model is examined for the case when the dependence structure is modelled by the FGM copula function and the marginal lifetime distribution of components conforms to a Weibull distribution. Further, a sensitivity analysis is performed to examine some important features of the model's parameters. We will see the unified framework developed not only generalizes age replacement policy and other classic maintenance models, but also allows considerable flexibility such that different scenarios can be explored.

Keywords: Inspection; Maintenance; Replacement; Hidden failures; Copula function; Renewal-reward theorem.

1. Introduction

Motivated by the extension of the approach in previous works, this paper is different with regard to presenting a unified copula-based maintenance model in which the decision process is driven by the excursion of the underlying state as a decision variable and the degradation phenomenon is modelled by the FGM copula function. The copula framework allows us to model the dependence structure among components of the system. The approach proposed here is typically appropriate for multi-component parallel systems with non-self-announcing failures and dependent components. Fire detectors, nuclear reactor safety systems, emergency core cooling systems, and protective devise are some examples of parallel systems whose failures are detected only by inspections. A common characteristic of such systems is that the failure of a component may not make its system fail and the system's failures are detected only by inspections. Therefore, repair and maintenance actions (see Hosseini, 2016; Fallahnezhad and Pourgharibshahi, 2017) including inspections and preventive replacements for such systems are essential to detect the system's failures and increase reliability and in turn availability (Kharazmi, 2017 and Javid et al., 2018) against risky and costly position arises from the system downtime.

In this paper, we address the maintenance scheduling problem with two tools. The common method of seeking the regeneration points is the first one. More specifically, the approach depends on the identification of an embedded renewal process defined by the replacement epochs and this allows the application of the renewal reward theorem and the formulation of the average cost rate used as a measure of policy. Secondly, the problem is addressed by considering a state process as a decision variable and partitioning the state space into two exclusive sets by means of a replacement threshold. In this way, the system is inspected to revel the true state of the system and maintenance actions are carried out on the basis of the observed system state falling into exclusive sets. Since both the maintenance costs and the level of maintenance are driven by the inter-inspection time and the replacement threshold, this raises an intriguing question that how often to inspect the system and when to replace the system that balances the two factors. This paper aims to answer these questions.

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Essentially, our model is an extension of optimal inspection policies and other maintenance policies whose attention is restricted to inspection and perfect repair on failure (corrective replacement). The most common policies for such systems include optimal inspection policies. It has a key role in maintenance cost and functioning systems. During the past decades, many optimum inspection policies as an extension of classical optimum checking policies (e.g. see Barlow et al. (1963); Hauge, (2002); Keller, (1974); Munford and Shahani, (1972)) for such systems have been studied. For example, Jiang and Jardine (2005) propose two optimization models to determine the optimum sequence of inspection times. In comparison to classical optimum inspection policies, their model more accurate and computationally simple. Chelbi et al. (2008) study an optimal inspection model for systems with self-announcing and non-self-announcing failures. The approach aims at determining the age T for inspection which maximizes the stationary availability of the system. Taghipour and Kassaei (2016) present a maintenance model to find the optimal inspection interval for a k-out-of-n load-sharing system. Chosen on the basis of the components state (total number of failed components), the decision maker's action is restricted to only minimal repair and perfect repair on failure. Rezaei (2017) proposes a maintenance model for inspection planning. Similar to Taghipour and Kassaei (2016), his model accounts two repair actions (minimal repair and perfect repair) for systems with failure-dependent components. Liu et al. (2017) schedule inspection intervals for multi-component systems characterized by hidden failures and dependent components. Recently, Seyedhosseini et al. (2018) propose an imperfect inspection model to find the optimal periodic inspection interval.

To develop the previous maintenance modelling approach, recent works have turned their attention to jointly determine optimal inspection interval and preventive replacement policy. For instance, He et al. (2015) propose a periodic inspection and preventive replacement policy for a system subject to hidden failures. Preventive replacement (PR) policy is implemented whenever the number of inspections scheduled between PRs reaches the quantity n. Babishin and Taghipour (2016) propose a joint optimal inspection and replacement policy for a k-out-of system. The optimal number of minimal repairs is used as a basis for maintenance decision making. Ahmadi and Wu (2017) given partial information propose a new approach for inspection scheduling and threshold-type replacement policy of parallel systems subject to hidden failures. Their approach rests on estimating a disruption time at which the total number of failed components reaches d. Recently, Ahmadi (2019) through the virtual age concept (e.g. see Lugtigheid et al. (2004); Kijima (1989)) develops extension of the existing modeling techniques from non-repairable systems (e.g. see Najari et al. (2018)) to repairable systems.

Additional motivation comes from the fact that dependencies of structure among components of the system are often neglected. Although some maintenance models examine the joint inspection and preventive replacement policy (Ahmadi and Wu, (2017, 2018); He et al., (2015); Babishin and Taghipour, (2016); Ahmadi (2019)), or regard the failure interaction of components (Taghipour and Kassaei, (2016); Liu, et al. (2017); Rezaei, (2017)), to our knowledge, a maintenance model which considers both factors does not exist. This paper aims to respectively develop the above modelling approach via (i) setting the model in an FGM copula framework and (ii) constructing a decision making process through using a state process as a decision variable. We will see the approach provides considerable flexibility in developing maintenance policies as well as matching the model to the real situations. In addition, the settings and the structure developed here allow different general repair models to be explored.

This paper shares some features with the works cited above, but it includes and investigates in a unifying framework some characteristics which have not been addressed or previously studied in isolation. More specifically, common and distinctive features of our model are as follows:

- Unlike most models, using a copula modeling technique, our modeling approach accounts for failure interaction among components. In this sense, our model is similar to the one suggested by Taghipour and Kasaei (2016); Liu, et al. (2017); Rezaei, (2017).
- In comparison to classical maintenance models cited above, our model differs in the action space including three kinds of actions: (i) no action, (ii) preventive replacement and (iii) corrective replacement.
- Our model resembles those models cited above considering the joint inspection and preventive replacement policy.
- The maintenance approach adopted in this paper is similar to that used by Ahmadi (2019); the decision process is driven by the excursions of a stochastic process $X(t)$ (decision variable) counting the total number of failed components. However, our model accounts for the failure interactions among components by means of an FGM copula function.
- Similar to Babishin and Taghipour (2016), He et al. (2015) and Ahmadi and Wu (2017), we consider a threshold-type policy, but our model differs in the decision variable; they respectively consider age, the number of inspections and the number of minimal repairs as decision variables.
- With the same approach as Lienhardt et al. (2008); Rasay et al. (2018); Hosseini et al. (2019) and Ahmadi (2019), the average cost rate is used as a measure of policy.

Furthermore, the most important contribution of the proposed model is that by means of the renewal reward theorem we make no assumptions about the process and the maintenance formulation is not restricted to some specific framework such as semi-Markov decision process. Additionally, the modeling approach explored here allows more general situations to be explored in later work.

2. Problem description and notations

We consider the problem of inspecting and maintaining an n-component parallel system characterized by non-self announcing failures and dependent components. The degradation process is modelled by a multi-dimensional FGM copula function. To monitor the system state and take an appropriate action, the system is inspected according to a periodic policy Π={$\tau, 2\tau, \cdots$} similar to that used by some researchers (e.g. see Tang et al. (2013); He et al. (2015); Rezaei, (2017)). With the same approach as Taghipour and Kassaei (2016), the components state is used as a basis for making maintenance decisions. More precisely, corrective and preventive maintenance actions carried out after an inspection are completely determined by the state of the stochastic process $X(t)$ counting the number of failed components up to age t. The decision process is steered by the excursions of the state process $\mathrm{X(t)} \in \Omega = \{x, x+1, \cdots, n\}$ where $X(0) = x$. Decisions can then be made by partitioning the state space Ω into exclusive subsets $A_0(j) = \{x, x+1, \cdots, j-1\}$, $A_1(j) = \{j, j+1, \cdots, n-1\}$ and the failure set $A_2 = \{n\}$. An inspection reveals the system state $X(t) \in A_i (i = 0,1,2)$, then an action is chosen on the basis of the set A_i. The action space denoted by the doubleton ‹a,3›={a_0, a_1, a_2} includes three kinds of actions: (i) no action, $\{a_0\}$, if the revealed state on inspection $X(t)$ falls in the state set $A_0(j)$, (ii) preventive replacement action, $\{a_1\}$, whenever on inspection the total number of failed components falls in the state set $A_1(j)$, and (iii) replace on failure, $\{a_2\}$, if on inspection $X(t)$ is found in the set $A_2 = \{n\}$. The structure of the model allows two variants of repair models to be examined; an age replacement model and a repair model whose action space is restricted to no action and perfect repair (corrective replacement). They are recovered by an appropriate choice of the decision threshold$j$. It is common that changes in the period of inspection τ and the preventive replacement threshold j may cause changes in the amount of maintenance and maintenance costs. On the one hand, inadequate maintenance may save maintenance cost but may result in undetected failures. On the other hand, excessive maintenance leads to increasing availability and detecting the system failure more rapidly, but it incurs higher maintenance costs. Thus, an optimal policy determined by (τ^*, j^*) is required to balance the amount of maintenance to increase availability against the costly and risky position arising from the system downtime. To this end, using the standard renewal theory argument, this paper minimizes the average cost rate for optimizing maintenance policy.

The paper is organized as follows. Section 2 includes the assumption and the degradation features of the model. The section is developed by giving some structural results. The maintenance model is described in Section 3. Section 4 formulates both the expected cost per cycle and the expected cycle length. The next section proposes a recursive algorithm to solve them. Section 6 demonstrates the generality of the proposed model and indicates how some current maintenance models emerge as specific cases. Some numerical results along with sensitivity analysis are given in section 7. Finally, the last section concludes the paper with a summary of the proposed model as well as future directions.

Notations

n:	The number of components of the system
C^*:	Copula function
θ:	Dependence degree
‹a,i›	The action space including i action(s)
Ω:	The state space
j:	Preventive replacement threshold
$A_0(j)$:	The subset of Ω associated with no action $\{a_0\}$
$A_1(j)$:	The subset of Ω associated with preventive replacement $\{a_1\}$
A_2:	The subset of Ω associated with corrective replacement $\{a_n\}$
τ:	Inspection interval
$\mathrm{T_i}$:	The lifetime of the ith component
$\mathrm{H_j^x}$:	The first hitting time of the set $A_1(j)$ by the state process $X(t)$
(α, β):	The shape and scale parameter of a Weibull degradation model
$X(t)$:	System state at age t
$X(0) = x$:	Starting state
$X_i(t)$:	The state of ith component
$\mu(t; x)$:	Mean past lifetime of the parallel system at age t
$P_{xu}(t)$:	The transition probability of $X(t)$ from x to u at age t
$F(t)$:	Common failure distribution of components
C_0:	Inspection cost
C_F:	Penalty cost incurred due to undetected failures
C_r:	Preventive replacement cost
C_R:	Corrective replacement cost
$\mathcal{C}_\tau(j; x)$:	Expected cost per cycle
$\ell_\tau(j; x)$:	Expected cycle length
$\mathbb{C}_\tau(j; x)$:	Expected cost per unit time
ϕ:	The Cumulative distribution of the standard normal distribution

3. Modeling the system

3.1. Assumptions

- The system is composed of n dependent components connected in parallel.
- The dependence structure of components is modelled by an FGM copula function.
- The marginal lifetime distribution of components is described by a Weibull distribution.
- Inspections are scheduled at fixed age intervals $\{\tau, 2\tau, \cdots\}$.
- Inspections are perfect and reveal the true state of components. This makes the state process $X(t)$ (the total number of failed components) discernible at fixed age intervals $\{\tau, 2\tau, \cdots\}$.
- Corrective and preventive maintenance actions are carried out in response to the system state revealed at $\{\tau, 2\tau, \cdots\}$.
- The preventive replacement is a threshold-type policy with respect to the state process $X(t)$.

3.2. Degradation model

Consider a parallel system consisting of n components with lifetimes T_i $(i = 1,2,\cdots,n)$ and corresponding common distribution function $F(t)$. The joint distribution function of random lifetimes T_i is specified by a multi-variate FGM copula function C^* with the dependence degree $\theta \in [0,1]$:

$$\mathrm{C}^*(\mathrm{u}_1, \mathrm{u}_2, \cdots, \mathrm{u}_\mathrm{n}) = \prod_{i=1}^{n} u_i + \theta \prod_{i=1}^{n} u_i(1-u_i); \quad \forall (u_1, u_2, \cdots u_n) \in I^n := [0,1]^n.$$

So, by setting $u_i = F(t_i)$ $(i = 1,2,\cdots,n)$, the joint lifetime distribution of components is given by

$$\mathrm{F}(\mathrm{t}_1, \mathrm{t}_2, \cdots, \mathrm{t}_\mathrm{n}) = \mathrm{C}^*\big(F(t_1), F(t_2), \cdots, F(t_n)\big) = \prod_{i=1}^{n} F(t_i) + \theta \prod_{i=1}^{n} F(t_i)(1-F(t_i)). \tag{1}$$

The model (1) is an extension of the trivariate FGM copula function studied by Spanhel and Kurz (2016). The model is developed by presenting a stochastic process $X_i(t)$ (i=1,2,$\cdots,n$) taking 0 or 1 if the component at age t is in a functioning state or failed state, respectively. In other words,

$$X_i(t) = \begin{cases} 0 & T_i > t \\ 1 & T_i \le t. \end{cases}$$

Thus, if $X(t)$ denotes the total number of failed component up to age t, then the counting process $X(t)$ (describing the system state) in terms of $X_i(t)$ can be expressed as

$$X(t) = \sum_{i=1}^{n} X_i\,(t).$$

Before proceeding to the next section, using the proposed degradation model, we study some reliability characteristics of the model. We will see how these measures contribute to the model development. For this, we assume that the lifetime of components conforms to a Weibull distribution with the shape and scale parameter (α, β):

$\bar{F}(t) = e^{-(t/\beta)^\alpha}$.

3.2.1. Mean Past Lifetime

Let $\mu(t;x)$ denote the mean past lifetime (MPL) of the system at age t given that the system starts operating at t = 0 with $(n-x)$ components, i.e. $X(0) = x$. Then given $\alpha = 2$ we have

$$\mu(\mathrm{t};\mathrm{x}) = \mathrm{E}(\mathrm{t} - \mathrm{T}|\mathrm{t} > \mathrm{T}) = \frac{\int_0^{\mathrm{t}} \mathrm{F}_{\mathrm{x:n}}(\mathrm{u})\mathrm{du}}{\mathrm{F}_{\mathrm{x:n}}(\mathrm{t})} = \frac{\mathbb{A}_1(\mathrm{t};\mathrm{x})}{\mathbb{A}_2(\mathrm{t};\mathrm{x})}, \tag{2}$$

where $F_{x:n}(t)$ denotes the lifetime distribution of an (n-x) component parallel system with random lifetime $T_{x:n}$,

$$\mathbb{A}_1(t;x) = \sum_{i=1}^{n-x} \binom{n-x}{i} (-1)^i \beta \sqrt{\frac{\pi}{i}} \times \left[\phi\left(\frac{\sqrt{2i}}{\beta} t\right) - 0.5\right] + \left[t + \theta\beta\sqrt{\frac{\pi}{n-x}} \times \left[\phi\left(\frac{\sqrt{2(n-x)}}{\beta} t\right) - 0.5\right]\right] +$$
$$\theta \sum_{i=1}^{n-x} \binom{n-x}{i} (-1)^i \beta \sqrt{\left(\frac{\pi}{n-x+i}\right)} \times \left[\phi\left(\frac{\sqrt{2(n-x+i)}}{\beta} t\right) - 0.5\right],$$

and

$$\mathbb{A}_2(t;x) = \left(1 - \exp(-(t/\beta)^2)\right)^{n-x} \times \left(1 + \theta \times \exp(-(n-x)(t/\beta)^2)\right).$$

Figure 1 demonstrates the behavior of the mean past lifetime of a 2-component system as a function of τ for different$\theta \in \{0.1, 0.9\}$. It indicates that the mean past lifetime of the system is an increasing function of the inspection intervalτ. In addition, the higher-level dependency $\theta: 0.1 \to 0.9$ makes the system more prone to failure. This may arise from the fact that in the absence of one component due to failure, a surplus load is transferred to another remaining component and so makes it more susceptible to failure.

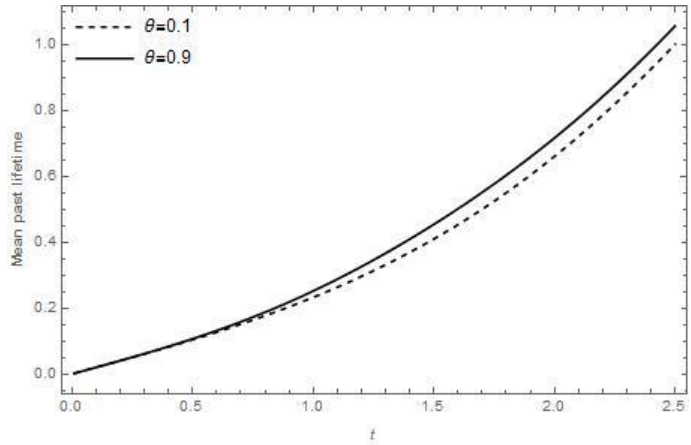

Figure 1. Mean past lifetime of the system given $(\boldsymbol{\alpha}, \boldsymbol{\beta}) = (\mathbf{2}, \sqrt{\mathbf{2}})$ and $(\boldsymbol{x}, \boldsymbol{n}) = (\mathbf{0}, \mathbf{2})$.

Proposition 1. Let $P_{xu}(t) = P(X(t) = u | X(0) = x)$ denote the transition probability of the system state from the starting state $X(0) = x$ to the state $X(t) = u$ at age t. Then given the assumption (1) we have

$$P_{xu}(t) = \mathfrak{B}_{\theta}\big(u - x; n - x, F(t)\big) = \mathfrak{B}\,(\mathrm{u} - \mathrm{x}; \mathrm{n} - \mathrm{x}, \mathrm{F(t)}) \times \left[1 + (-1)^{n-u}\tilde{\theta} \times \mathfrak{B}(u - x; n - x, \bar{F}(t))\right] \qquad (3)$$

where

$\tilde{\theta} = \theta \div \binom{n-x}{u-x}$,

and

$\mathfrak{B}(u - x; n - x, \bar{F}(t)) = \binom{n-x}{u-x} F(t)^{u-x} \bar{F}(t)^{n-u}$.

To help with intuition on the behavior of the system, as a function of the inspection period, an evolution of transition probabilities for $\theta \in \{0.1, 0.9\}$ as $(x, n) = (2,5)$ is given (see Figure 2).

4. Maintenance model

The decision process is directed by the excursions of the state process $X(t) \in \Omega$ split into three exclusive subsets $A_0(j) = \{x, x+1, \cdots, j-1\}$, $A_1(j) = \{j, j+1, \cdots, n-1\}$ and $A_2 = \{n\}$. Inspections reveal the true state of the system as $X(t) \in A_i$ $(i = 0,1,2)$ Then an action of the action space ‹a,3›=$\{a_0, a_1, a_2\}$ including no action, $\{a_0\}$, preventive replacement, $\{a_1\}$, and corrective replacement, $\{a_2\}$ is chosen on the basis of the set A_i. In fact, the decision maker inspects the system according to a periodic policy Π=$\{k\tau: k = 1,2, \cdots\}$. Inspections reveal the true state of the system and preventive and corrective maintenance actions are carried out in response to the observed system state.

Preventive actions are decided by partitioning the state space Ω into sets $A_0(j)$ and $A_1(j)$ where the decision threshold j is used as the definition of preventive replacement action: as an observation indicates that $X(t) \in A_0(j) = \{x, x+1, \cdots, j-1\}$, the system is not repaired until the next inspection (no action). This action which incurs a cost C_0 is denoted by $\{a_0\}$. On making an inspection if $X(t) \in A_1(j) = \{j, j+1, \cdots, n-1\}$ preventive replacement actions, $\{a_1\}$, are taken with corresponding cost C_r. The system is regarded as failed and a subsequent corrective maintenance action (replacement), $\{a_n\}$, with cost C_R $(C_R > C_r)$ is performed if on inspection $X(t) \in A_n = \{n\}$.

The above threshold-type replacement policy raises an intriguing question that how to get an estimate of the minimum number of inspections until the system is observed in set $A_1(j)$. We formulate the answer to this question as the following proposition.

Proposition 2. Let the starting state of the system be $X(0) = x$ and H_j^x denote the first hitting time of the set $A_1(j)$ by the state process $X(t)$:

$\mathrm{H}_{\mathrm{j}}^{\mathrm{x}} = \inf\{\mathrm{t} \in \mathbb{R}_+ : X(t) \in A_1(j)\}$.

Then the minimum number of inspection until the system is observed in $A_1(j)$ is

$$\mathrm{n}_0 = \left\lfloor \frac{\mu_{j:n}}{\tau} \right\rfloor + 1, \qquad (4)$$

where $\mu_{j:n}$ denotes the mean hitting time of $A_1(j)$ by the state process expressed as

$$\mu_{j:n} = \mathbb{E}\left(H_j^x\right) = \beta\Gamma(1+1/\alpha) \times \left[\sum_{k=0}^{j-1}\sum_{i=0}^{k}\binom{n_x}{k}\binom{k}{i}\frac{(-1)^i}{(n_x+i-k)^{1/\alpha}} + \theta\sum_{k=0}^{j-1}\sum_{i=0}^{n_x}\binom{n_x}{k}\binom{n_x}{i}\frac{(-1)^{n_x+i-k}}{(n_x+i)^{1/\alpha}}\right], \quad (5)$$

and $n_x = n - x$.

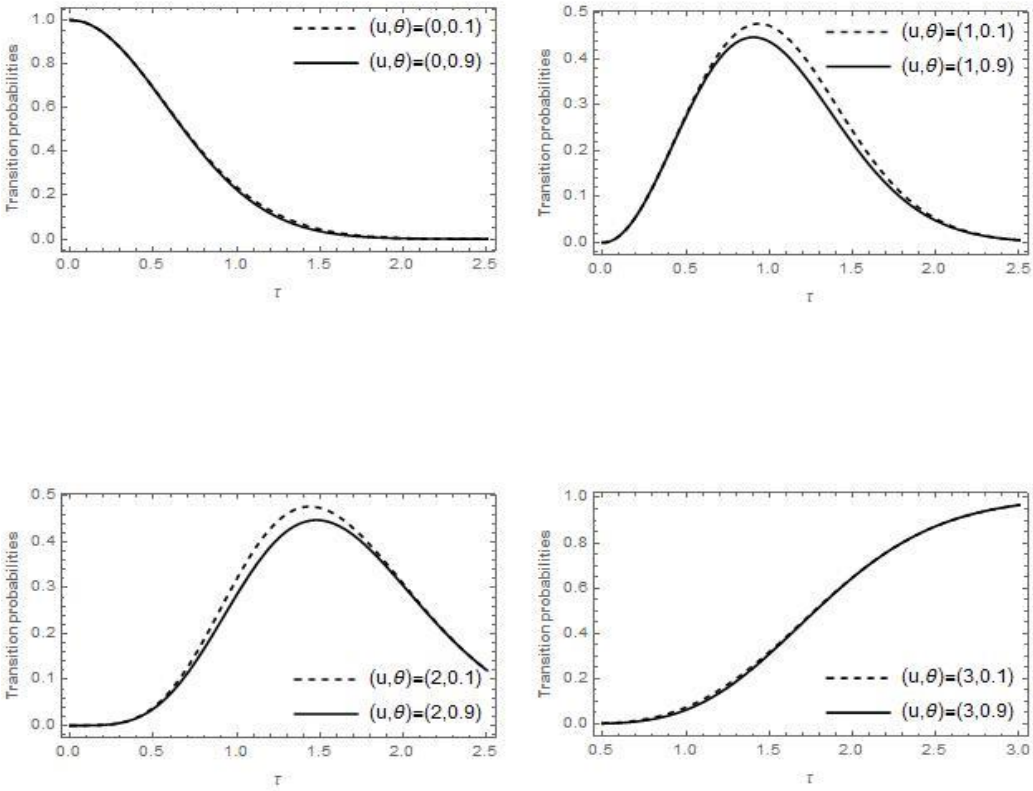

Figure 2. Transition probabilities $\boldsymbol{P_{xu}(\tau)}$ as a function of $\boldsymbol{\tau}$ for different $\boldsymbol{\theta} \in \{\mathbf{0.1}, \mathbf{0.9}\}$ as $(\boldsymbol{x}, \boldsymbol{n}) = (\mathbf{2}, \mathbf{5})$.

Proof. From (1) we get

$$P\left(H_j^x > t\right) = \sum_{k=0}^{j-1}\binom{n}{k}F(t)^k\,\bar{F}(t)^{n_x-i} \times [1+\theta(-1)^{n_x-i}\bar{F}(t)^k F(t)^{n_x-k}]. \quad (6)$$

Since

$$\left(1-\exp(-(t/\beta)^\alpha)\right)^u = \sum_{i=0}^{u}\binom{u}{i}(-1)^i\exp(-i(t/\beta)^\alpha) \quad (7)$$

by plugging (7) in (6) we have

$$P\left(H_j^x > t\right) = \sum_{k=0}^{j-1}\sum_{i=0}^{k}\binom{n_x}{k}\binom{k}{i}(-1)^i\exp[-(n_x+i-k)(t/\beta)^\alpha] + \theta\sum_{k=0}^{j-1}\sum_{i=0}^{n_x}\binom{n_x}{k}\binom{n_x}{i}(-1)^{n_x+i-k}\exp[-(n_x + i)(t/\beta)^\alpha]. \quad (8)$$

By integrating both sides of the above equation, the results are proved.□

Figure 3 illustrates that the hitting time distribution of the set $A_1(j)$ by the state process $X(t)$ is a decreasing function of the preventive replacement threshold j. This intuitively implies that a lower threshold results in an earlier preventive replacement (see Table 3). The results are given known parameters $(0) = 0$, $(\alpha, \beta) = (2, \sqrt{5})$ and $(n, \theta) = (5, 0.6)$.

Table 1. Mean hitting time for $\boldsymbol{j} \in \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$

j	1	2	3	4
$\mu_{j:n}$	1.41	1.88	2.45	3.26

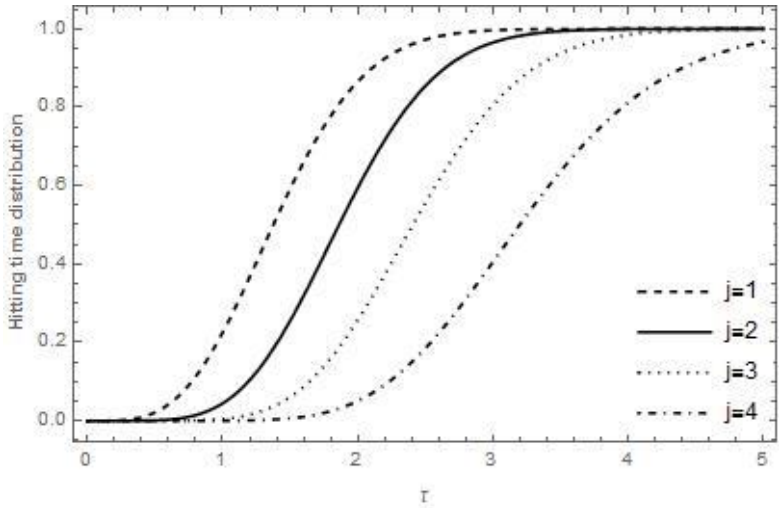

Figure 3. Hitting time distribution for different preventive replacement threshold $\boldsymbol{j}$

5. Minimizing the Long-Run Average Costs

Since the maintenance costs and the level of maintenance are driven by the inspection interval τ and the replacement threshold j, the objective of the paper is to determine an optimal inspection and replacement policy which truly balance both factors. For this, the long-run average cost rate is used as a measure of policy. Because the replacement instants are regeneration points, the time between two consecutive replacements is a renewal cycle. Therefore, by the standard renewal theory arguments, the long-run average cost rate can be derived by the expected cost per cycle divided by the expected cycle length.

5.1. Expected cost per cycle

A cycle consists of a sequence of inspections and maintenance actions that ends with planned or unplanned replacement. Corrective and preventive maintenance actions costs incurred in a cycle are random. Let C_τ^x denote the cost per cycle given starting state $X(0) = x$, that means the system starts operating with $(n - x)$ components. At inspection time τ if the system state is observed in $A_0(j)$ the system restarts from the current state $X(\tau) \in A_0(j)$. It incurs the planned inspection cost C_0 and the future costs starting in state $X(\tau)$. An additional cost would be incurred if on inspection the revealed state falls in $A_1(j)$. In this case the system is returned to the perfect working state (preventive replacement) with the replacement cost C_r and the future cost starting in state $X(0) = 0$. If the system is found in a failed state it undergoes a corrective maintenance. It incurs an unplanned replacement cost C_R $(C_R > C_r)$ and a penalty cost per unit time C_F due to an undetected failure within inter-inspection times. In other words,

$$C_\tau^x = \left(C_0 + C_\tau^{X(\tau)}\right) I\left(X(\tau) \in A_0(j)\right) + \left(C_r + C_\tau^0\right) I\left(X(\tau) \in A_1(j)\right) + \left(C_R + C_F(\tau - T_{x:n})\right) I(X(\tau) \in A_n) \tag{9}$$

where C_τ^0 arises from the preventive replacement which resets all processes to zero. Taking expectations of both sides of (9) gives the expected cost per cycle $\mathcal{C}_\tau(j; x) = E(C_\tau^x)$:

$$\mathcal{C}_\tau(\mathrm{j}; \mathrm{x}) = \sum_{\mathrm{u}=\mathrm{x}}^{\mathrm{j}-1} \left(\mathrm{C}_0 + \mathcal{C}_\tau(\mathrm{j}; \mathrm{u})\right) \mathfrak{B}_\theta\left(\mathrm{u} - \mathrm{x}; \mathrm{n} - \mathrm{x}, \mathrm{F}(\tau)\right) + \left(\mathrm{C}_\mathrm{r} + \mathcal{C}_\mathrm{r}(\mathrm{j}; 0)\right) \sum_{\mathrm{u}=\mathrm{j}_\mathrm{x}}^{\mathrm{n}-1} \mathfrak{B}_\theta\left(\mathrm{u} - \mathrm{x}; \mathrm{n} - \mathrm{x}, \mathrm{F}(\tau)\right) + \mathrm{F}_{\mathrm{x:n}}(\tau)\left(\mathrm{C}_\mathrm{R} + \mathrm{C}_\mathrm{F}\mu(\tau; \mathrm{n} - \mathrm{x})\right) \tag{10}$$

where $\mu(\tau; \mathrm{n} - \mathrm{x})$ and $\mathfrak{B}_\theta\left(\mathrm{u} - \mathrm{x}; \mathrm{n} - \mathrm{x}, \mathrm{F}(\tau)\right)$ are given in (2) and (3) respectively and $\mathrm{j}_\mathrm{x} = \mathrm{Max}(\mathrm{x}, \mathrm{j})$.

5.2. Expected cycle length

Let L_τ^x denote the cycle length starting in $X(0) = x$. Using the same argument as above the expected cycle length $\ell_\tau(j;x) = E(L_\tau^x)$ is obtained: if at inspection time the system is observed in $A_0(j)$ the cycle length consists of an inspection time and an additional cycle length starting from $X(\tau)$. In the perfect repair case, when finding the system in $A_1(j)$ the cycle length is made up of a full period τ and an additional random time L_τ^0 starting in state $X\ (0) = x$. On failure at τ the cycle length is completed. In other words,

$$\mathrm{L}_\tau^{\mathrm{x}} = \left(\tau + \mathrm{L}_\tau^{\mathrm{X}(\tau)}\right)\mathrm{I}(\mathrm{X}(\tau) \in \mathrm{A}_0(\mathrm{j})) + (\tau + \mathrm{L}_\tau^0)\mathrm{I}(\mathrm{X}(\tau) \in \mathrm{A}_1(\mathrm{j})) + \tau\mathrm{I}(\mathrm{X}(\tau) \in \mathrm{A}_\mathrm{n}). \tag{11}$$

Taking expectations of both sides of equation (11) gives the expected cycle length $\ell_\tau(\mathrm{j};\mathrm{x}) = \mathbb{E}(\mathrm{L}_\tau^{\mathrm{x}})$:

$$(\mathrm{j};\mathrm{x}) = \tau + \sum_{\mathrm{u}=\mathrm{x}}^{\mathrm{j}-1} \ell_\tau(\mathrm{j};\mathrm{u})\mathfrak{B}_\theta\big(\mathrm{u}-\mathrm{x};\mathrm{n}-\mathrm{x},\mathrm{F}(\tau)\big) + \ell_\tau\,(\mathrm{j};0)\sum_{\mathrm{u}=\mathrm{j}_\mathrm{x}}^{\mathrm{n}-1}\mathfrak{B}_\theta\big(\mathrm{u}-\mathrm{x};\mathrm{n}-\mathrm{x},\mathrm{F}(\tau)\big). \tag{12}$$

Thus, using the equations (10) and (12), the average cost rate can be given by

$$\mathbb{C}_\tau(\mathrm{j};\mathrm{x}) = \frac{\mathcal{C}_\tau(\mathrm{j};\mathrm{x})}{\ell_\tau(\mathrm{j};\mathrm{x})}. \tag{13}$$

The optimal period of inspection and preventive maintenance thresholds (τ^*, j^*) can then be determined as:

$$(\tau^*, j^*) = \underset{(\tau,j)\in(0,\infty)\times\Omega}{\mathrm{Argmin}}\ \mathbb{C}_\tau(j;x). \tag{14}$$

6. Recursive scheme

In this section, we get a general solution (13) to solve problem (14). To this end, we devise a recursive algorithm to solve for $\mathcal{C}_\tau(\mathrm{j};\mathrm{x})$ and $\ell_\tau(j;x)$. To facilitate the presentation, let $\mathcal{C}_\tau^{(\mathrm{i})}(\mathrm{j};\mathrm{x})$ and $\ell_\tau^{(\mathrm{i})}(\mathrm{j};\mathrm{x})$ $(\mathrm{i} = 1,2,3)$ be expressed as

$$\mathcal{C}_\tau^{(1)}(\mathrm{j};\mathrm{x}) = C_0\sum_{u=x}^{j-1}\mathfrak{B}_\theta\big(\mathrm{u}-\mathrm{x};\mathrm{n}-\mathrm{x},\mathrm{F}(\tau)\big) + C_r\sum_{u=j_x}^{n-1}\mathfrak{B}_\theta\big(\mathrm{u}-\mathrm{x};\mathrm{n}-\mathrm{x},\mathrm{F}(\tau)\big) + F_{x:n}(\tau)\big(C_R + C_F\mu(\tau;n-x)\big),$$

$$\mathcal{C}_\tau^{(2)}(j;x) = \sum_{u=x+1}^{j-1}\mathcal{C}_\tau\,(j;u)\mathfrak{B}_\theta\big(\mathrm{u}-\mathrm{x};\mathrm{n}-\mathrm{x},\mathrm{F}(\tau)\big) + \mathcal{C}_\tau(j;0)\sum_{u=j_x}^{n-1}\mathfrak{B}_\theta\big(\mathrm{u}-\mathrm{x};\mathrm{n}-\mathrm{x},\mathrm{F}(\tau)\big),$$

$$\mathcal{C}_\tau^{(3)}(j;x) = 1 - \mathfrak{B}_\theta\big(0;\mathrm{n}-\mathrm{x},\mathrm{F}(\tau)\big)I(x \le j-1).$$

In addition,

$$\ell_\tau^{(1)}(j;x) = \sum_{ux+1}^{j-1}\ell_\tau\,(j;u)\mathfrak{B}_\theta\big(\mathrm{u}-\mathrm{x};\mathrm{n}-\mathrm{x},\mathrm{F}(\tau)\big),$$

$$\ell_\tau^{(2)}(j;x) = \ell_\tau(j;0)\sum_{u=j_x}^{n-1}\mathfrak{B}_\theta\big(\mathrm{u}-\mathrm{x};\mathrm{n}-\mathrm{x},\mathrm{F}(\tau)\big),$$

$$\ell_\tau^{(3)}(j;x) = 1 - \mathfrak{B}_\theta\big(0;\mathrm{n}-\mathrm{x},\mathrm{F}(\tau)\big)I(x \le j-1).$$

Then, given starting state $X(0) = x$ both the expected cost per cycle $\mathcal{C}_\tau(j;x)$ and the expected cycle length $\ell_\tau(j;x)$ can be solved recursively for $x = n-1,\cdots,1,0$:

$$\mathcal{C}_\tau(j;x) = \frac{\mathcal{C}_\tau^{(1)}(\mathrm{j};\mathrm{x}) + \mathcal{C}_\tau^{(2)}(\mathrm{j};\mathrm{x})}{\mathcal{C}_\tau^{(3)}(\mathrm{j};\mathrm{x})},$$

and

$$\ell_\tau(\mathrm{j};\mathrm{x}) = \frac{\tau + \ell_\tau^{(1)}(\mathrm{j};\mathrm{x}) + \ell_\tau^{(2)}(\mathrm{j};\mathrm{x})}{\ell_\tau^{(3)}(\mathrm{j};\mathrm{x})}.$$

7. Specific models

The model above encompasses some variants of the repair models as special cases. They can be retrieved by an appropriate choice of the maintenance threshold j. The specific models are given in the following.

7.1. Variant 1 repair model: ‹a,2›={$\boldsymbol{a_0}, \boldsymbol{a_n}$}

Letting $j = n$ recover a special case of our explored model with two possible actions at each decision instant: no action if the state is observed in $A_0(j) = \{x, x+1, \cdots, n-1\}$; otherwise it undergoes a major repair at cost C_R. In this case, (10) and (12) become

$$\mathcal{C}_\tau(\mathrm{j};\mathrm{x}) = \sum_{\mathrm{u}=\mathrm{x}}^{\mathrm{n}-1}\big(\mathrm{C}_0 + \mathcal{C}_\tau(\mathrm{j};\mathrm{u})\big)\,\mathfrak{B}_\theta\big(\mathrm{u}-\mathrm{x};\mathrm{n}-\mathrm{x},\mathrm{F}(\tau)\big) + F_{x:n}(\tau)\big(C_R + C_F\mu(\tau;n-x)\big), \tag{15}$$

And

$$\ell_\tau(\mathrm{j};\mathrm{x}) = \tau + \sum_{\mathrm{u}=\mathrm{x}}^{\mathrm{n}-1} \ell_\tau(\mathrm{j};\mathrm{u})\mathfrak{B}_\theta\big(\mathrm{u}-\mathrm{x};\mathrm{n}-\mathrm{x},\mathrm{F}(\tau)\big). \tag{16}$$

7.2. Variant 2 repair model: ‹a,2›={a_1, a_n}

For variant 2 recovered by choosing $j = x$, there are two possible actions at each decision instant: (i) planned preventive replacement if the system state falls in the set $A_1(j)$, and (ii) corrective replacement if the system state is found in a failed state. Given the above assumption,

$$\mathcal{C}_\tau(\mathrm{j};\mathrm{x}) = \big(C_r + \mathcal{C}_\tau(j;0)\big)\sum_{u=x}^{n-1}\mathfrak{B}_\theta\big(\mathrm{u}-\mathrm{x};\mathrm{n}-\mathrm{x},\mathrm{F}(\tau)\big) + F_{x:n}(\tau)\,\big(C_R + C_F\mu(\tau;n-x)\big), \tag{17}$$

and

$$\ell_\tau(j;x) = \tau + \ell_\tau(j;0)\sum_{u=x}^{n-1}\mathfrak{B}_\theta\big(\mathrm{u}-\mathrm{x};\mathrm{n}-\mathrm{x},\mathrm{F}(\tau)\big). \tag{18}$$

As noted, this is similar to the age replacement policy, but the preventive replacement cost at the end of a cycle depends on the system state $X(\tau)$ (the total number of failed components). A two-region replacement policy with a preventive replacement cost C_r if the system is in functioning state, and an unplanned replacement cost including C_R and C_F otherwise recreates the age replacement policy. More specifically, if $T_{x:n}$ denotes the lifetime of an $(n-x)$ -components system, (10) and (12) in terms of $T_{x:n}$ can be reformulated as

$$\mathcal{C}_\tau(\mathrm{j};\mathrm{x}) = \big(C_r + \mathcal{C}_\tau(j;0)\big) \times \bar{F}_{x:n}(\tau) + F_{x:n}(\tau)\big(C_R + C_F\mu(\tau;n-x)\big), \tag{19}$$

And

$$\ell_\tau(j;x) = \tau + \ell_\tau(j;0) \times \bar{F}_{x:n}(\tau). \tag{20}$$

As noted, since

$$\bar{F}_{x:n}(\tau) = \sum_{u=x}^{n-1} P_{xu}\,(\tau) = \sum_{u=x}^{n-1}\mathfrak{B}_\theta\big(\mathrm{u}-\mathrm{x};\mathrm{n}-\mathrm{x},\mathrm{F}(\tau)\big),$$

equations (19) and (20) respectively coincide with (17) and (18). This generalization, in comparison to an ordinary age replacement policy, permits degradation-dependent age replacement policies to be examined.

8. Numerical results

Consider an n-component parallel system whose failure mechanism is expressed by multidimensional FGM distribution (1) with the common marginal Weibull distribution function $F(t)$ given as above. For the numerical illustration, we set $n = 5$, $(\alpha,\beta) = \big(2,\sqrt{5}\big)$, $\theta = 0.6$ and $x = 0$. The latter means the system starts operating with five components. The cost parameters of the model are $C_0 = 1.4$, $C_r = 3.4.$, $C_R = 7$ and $C_F = 3$.

By using the recursive algorithm given in section 5, our aim is to minimize the average cost rate (13) with respect to maintenance parameters including the inspection interval τ and the preventive replacement threshold j. Using the above set of known values, optimal solutions are given in Table 2. One can see that given $C_F = 3$ and $\theta = 0.6$ the maintenance actions should be scheduled with respect to the optimal maintenance parameters (τ^*, j^*):

$$(\tau^*, j^*) = \underset{(\tau,j)\in(0,\infty)\times\Omega}{\mathrm{Argmin}}\ \mathbb{C}_\tau(j;x) = (2.1630,4). \tag{21}$$

This incurs the minimum maintenance cost $\mathbb{C}_{\tau^*}(j^*;0) = 1.54$. The structure of the optimal policy can be described as follows: the system should be inspected at periodic times $\Pi^* = \{\mathrm{k}\tau^*: \mathrm{k} = 1,2,\cdots\}$ with the optimal inspection period $\tau^* = 2.1630$. Inspections reveal the true state of components and this allows preventive and corrective maintenance actions to be carried out in response to the observed system state: if at $\tau^* = 2.1630$ the system state is found in $A_0(j^*) = \{0,1,2,3\}$ (at the most three components experience failure) no action is taken, if the system state is observed in $A_1(j^*) = \{4\}$ (four out of five components experience failure) the system is preventively replaced by a new one; otherwise the system undergoes corrective maintenance.

Table 2. Optimal solutions $\big(\boldsymbol{j}^*, \boldsymbol{\tau}^*, \mathbb{C}_{\boldsymbol{\tau}^*}(\mathbf{j}^*, \mathbf{0})\big)$ given $\boldsymbol{n} = \mathbf{5}$ for different dependence degrees $\boldsymbol{\theta}$ and penalty costs $\boldsymbol{C_F}$.

	Dependence degree: θ					
C_F	0	0.2	0.4	0.6	0.8	1.0
3	(4; 1:649; 1:49)	(4; 1:643; 1:51)	(4; 1:637; 1:52)	(4; 1:630; 1:54)	(4; 1:622; 1:55)	(4,1.613,1.57)
6	(4; 1:540; 1:54)	(4; 1:527; 2:55)	(4; 1:512; 2:57)	(3; 1:749; 2:59)	(3; 1:750; 1:59)	(3,1.752,1.60)
9	(4; 1:467; 1:57)	(4; 1:449; 1:59)	(3; 1:677; 1:60)	(3; 1:675; 1:61)	(3; 1:674; 1:62)	(3,1.673,1.62)
12	(4; 1:412; 1:60)	(3; 1:626; 1:61)	(3; 1:623; 1:62)	(3; 1:620; 1:63)	(3; 1:617; 1:64)	(3,1.613,1.64)

The results are developed by examining the response of optimal solutions to the penalty cost and the dependence degree with $C_F = \{3,6,9,12\}$ and $\theta \in \{0.2k: k = 1,2,\cdots,5\}$. The results reveal that as components operate independently $(\theta = 0)$, changing C_F does not affect the optimal repair threshold j^*, but on the other hand the optimal period of inspection decreases with C_F making inspections more frequent. The model postulates a lower optimal replacement

threshold $j^*: 4 \to 3$ as both the dependence degree and the penalty cost increase uniformly. In this case, the model penalizes a costly strategy which favors too many inspections. But, it is not the case as the optimal replacement threshold remains fixed. This favorably leads to an earlier detection of the system failure hence reducing the penalty cost. The behavior of the average cost rate is also presented graphically in Figure 4 which indicates that the repair model characterized by the optimal replacement threshold $j^* = 4$ outperforms other repair models including the age replacement model (see Section 6.2) and the variant 1 repair model whose action space is restricted to inspection and perfect repair, i.e. $\langle a, 2\rangle = \{a_0, a_n\}$.

Table 3 and Figure 5 demonstrate the effect of the starting state $X(0) = x$ on the model. The results reveal that as surviving components at initial time decrease (x increases), the system becomes more susceptible to failure. This induces a reduction in the cycle length and so an increase in resulting average cost rate, but on the other hand makes inspections more frequent.

Table 3. Optimal solutions $(\boldsymbol{\tau}^*, \mathbb{C}_{\boldsymbol{\tau}^*}(\boldsymbol{j}^*, \boldsymbol{x}))$ for different starting states $\boldsymbol{x} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$ given $\boldsymbol{j}^* = \mathbf{4}$ and $\boldsymbol{\theta} = \mathbf{0.6}$.

x	0	1	2	3	4
$(\tau^*, \mathbb{C}_{\tau^*}(j^*, x))$	(1.163,1.54)	(1.585,1.56)	(1.525,1.59)	(1.426,1.64)	(1.345,1.69)

Given $(0) = 0$, measured by the ratio of the average cost rate $\mathbb{C}_\tau(4; 0)$ to the average cost rate $\mathbb{C}_\tau(j; 0)$ $(j \neq 4)$, the cost efficiency of the optimal threshold policy $j^* = 4$, i.e. $e_\tau(4, j)$:

$$e_c(4; j) = \frac{\mathbb{C}_\tau(4; 0)}{\mathbb{C}_\tau(j; 0)},$$

as a function of the inspection interval τ is illustrated by Figure 6. It is evident that when $\mathbb{C}_\tau(4; 0) \div \mathbb{C}_\tau(j; 0) < 1$ $(j \neq 4)$, the threshold policy $j^* = 4$ is more cost efficient than other threshold policies. Figure 6 indicates that (i) the cost efficiency does not respond to the threshold value j as the inspection period is large enough $(\tau \in [4, \infty))$, (ii) the repair model corresponding to the threshold policy $j^* = 4$ is consistently more cost efficient than the variant 1 repair model $(j = 5)$, (iii) within the inspection intervals (2,4) and (2.6,4) the repair models with $j = 3$ and $j \in \{0,1,2\}$ respectively are slightly more cost efficient than the repair model with $j^* = 4$. As noted, for $\tau \notin (2.6,4)$ the threshold policy $j^* = 4$, particularly for small periods of inspection is more cost efficient than the age replacement policy $(e_c(4,0) < 1)$.

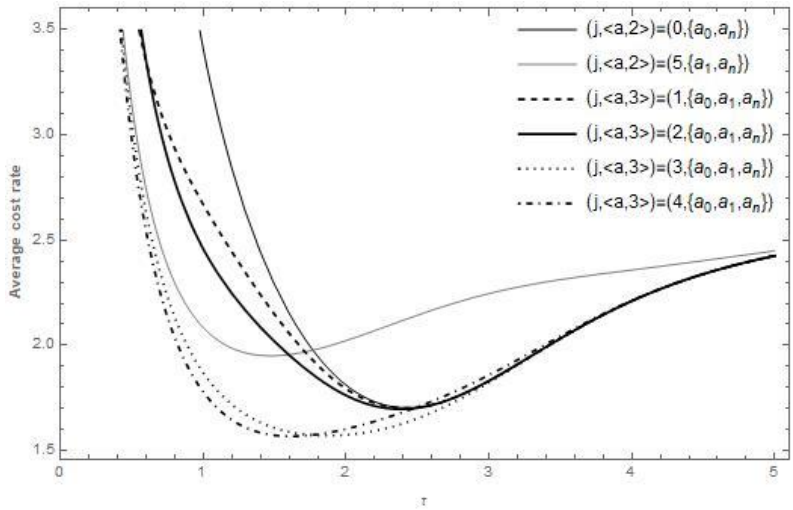

Figure 4 . Average cost rate for different preventive replacement thresholds $\boldsymbol{j} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ given starting state $\boldsymbol{x} = \mathbf{0}$ and the dependence degree $\boldsymbol{\theta} = \mathbf{0.6}$.

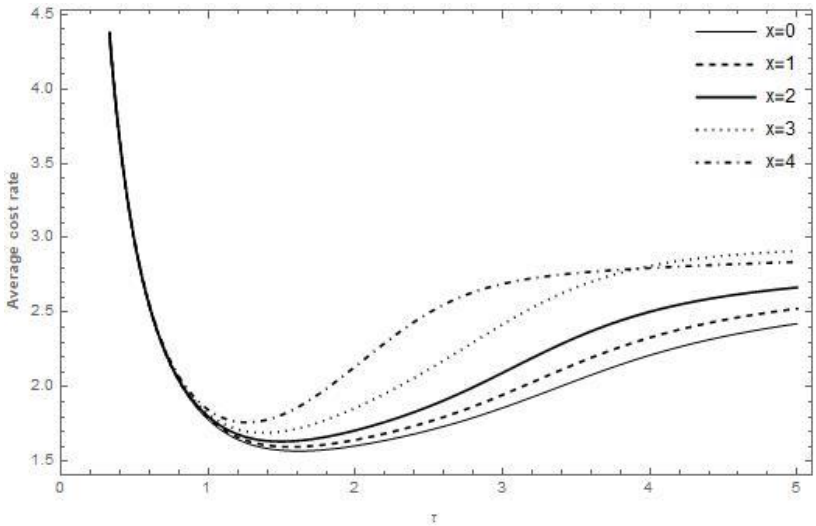

Figure 5. Average cost rate for different starting states $\boldsymbol{x} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}\}$ given optimal replacement threshold $\boldsymbol{j}^* = \mathbf{4}$ and the dependence degree $\boldsymbol{\theta} = \mathbf{0.6}$.

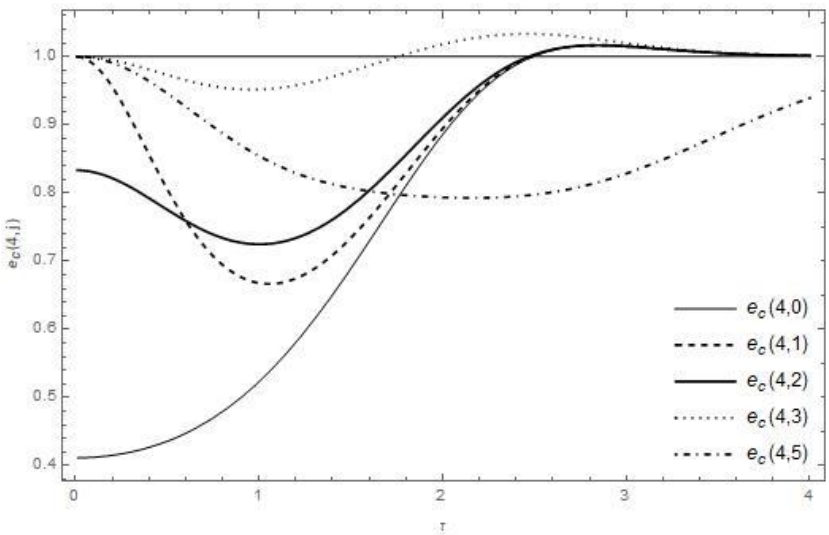

Figure 6. The cost efficiency $\boldsymbol{e_c(4;j)}$ as a function of $\boldsymbol{\tau}$ for different preventive replacement thresholds $\boldsymbol{j} \in \{\mathbf{0}, \mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{5}\}$ and the dependence degree $\boldsymbol{\theta} = \mathbf{0.6}$.

9. Conclusions

Using an FGM copula function, this paper has presented a unified maintenance model to jointly determine both optimal inspection intervals and optimal preventive replacement policy for parallel systems with non-self-announcing failures and dependent components. The model employs the renewal reward theorem based on the identification of an embedded renewal process. This allows for formulating the average cost rate used as a measure of policy to optimize the model with respect to maintenance parameters. The explored model generalizes age replacement models and outperforms those maintenance models whose attentions are restricted to inspection and perfect repair (corrective replacement), or failure dependencies between components are ignored.

The results of the model give sensible and realistic inspection and preventive replacement policies for parallel systems whose components lifetime conforms to a Weibull distribution. Also, the findings give insight into the behavior of the model as some parameters including the dependence factor, starting state and penalty cost vary. In short, the main findings of our numerical investigation are as follows:

- The optimal model is characterized by the optimal inspection interval $\tau^* = 2.163$ and the optimal replacement threshold $j^* = 4$.
- The optimal model with $j^* = 4$ is more cost efficient than the variant 1 repair model with $j = 5$.
- The maintenance approach adopted in our model outperforms age replacement and other simple maintenance strategies.
- Increasing the penalty cost makes inspections more frequent.

• The higher level of redundancy at initial time (increasing starting state) induces a decrease in the expected cycle length. This leads to an increase in both the number of inspections and the average cost rate.
• An increased interaction of failure characterized by the higher level of dependency leads to an increased tendency of failure and hence an earlier preventive replacement time (j^*decreases). The optimal replacement threshold j^* responds similarly to the penalty cost.
• In the absence of dependency ($\theta = 0$) the optimal replacement threshold j^* does not respond to the penalty cost (a risky position).

The model explored here indicates an approach which will be extended by setting the degradation model in a Levy copula framework. In comparison to an ordinary copula, the Levy copula framework allows for considerable flexibility to model the stochastic dependence for degradation processes of multi-component systems. Also, possible extension of practical interest includes the case of partial repairs (e.g. see Huynh, (2019); Huynh, (2020); Mercier and Castro, (2019); Syamsundara et al., (2020); Yang (2019) and Zhang, (2020)) implemented through partitioning the state space into four non-overlapping state sets and incorporating an age reduction model. The model here and the methodology in an earlier paper by Ahmadi (2019) show the possibility of this scheme. Further, the formulation could be developed from a one-dimensional process to multi-dimensional processes. In this case, a transform of the underlying process could be used as a decision variable. This structure makes the maintenance modeling of non-homogeneous populations by means of a multivariate process possible.

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